Canonical key formula for projective abelian schemes

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Abstract. In this paper we prove a refined version of the canonical key formula for projective abelian schemes in the sense of Moret-Bailly (cf. [MB]), we also extend this discussion to the context of Arakelov geometry. Precisely, let $\pi : A \to S$ be a projective abelian scheme over a locally noetherian scheme S with unit section $e : S \to A$ and let L be a symmetric, rigidified, relatively ample line bundle on A. Denote by ω_A the determinant of the sheaf of differentials of π and by d the rank of the locally free sheaf $\pi_* L$. In this paper, we shall prove the following results: (i). there is an isomorphism

$$\det(\pi_* L)^{\otimes 24} \cong (e^* \omega_A^\vee)^{\otimes 12d}$$

which is canonical in the sense that it is compatible with arbitrary base-change; (ii). if the generic fibre of S is separated and smooth, then there exist positive integer m, canonical metrics on L and on ω_A such that there exists an isometry

$$\det(\pi_* \overline{L})^{\otimes 2m} \cong (e^* \overline{\omega}_A^\vee)^{\otimes md}$$

which is canonical in the sense of (i). Here the constant m only depends on g, d and is independent of L.

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1 Introduction

Let $\pi : A \to S$ be a projective abelian scheme with unit section $e : S \to A$, where S is a normal excellent scheme. Let L be a symmetric, rigidified, relatively ample line bundle on A. It is well known that L is π-acyclic and $\pi_* L$ is a locally free coherent sheaf on S (cf. [MFK, Proposition 6.13]). We denote by d the rank of $\pi_* L$. Moreover, denote by ω_A the determinant of the sheaf of differentials of π. In this situation, Moret-Bailly proves that there exist a positive integer m and an isomorphism

$$\det(\pi_* L)^{\otimes 2m} \cong (e^* \omega_A^\vee)^{\otimes md}$$

of line bundles on S (cf. [MB, Appendice 2, 1.1]). If we write

$$\Delta(L) := \det(\pi_* L)^{\otimes 2} \otimes (e^* \omega_A)^{\otimes d},$$

then Moret-Bailly's result states that $\Delta(L)$ has a torsion class in the Picard group $\mathrm{Pic}(S)$. This is so called the key formula for projective abelian schemes, and it is denoted by $\mathrm{FC}^{\mathrm{ab}}(S, g, d)$.

When S is a scheme which is quasi-projective over an affine noetherian scheme and d is invertible on S, the fact that $\Delta(L)$ is a torsion line bundle is a consequence of the Grothendieck-Riemann-Roch theorem. This was shown by Moret-Bailly and Szpiro in [MB, Appendice 2, 1.3, 1.4] and also by Chai in his thesis [Chai1, V, Theorem 3.1, p. 209].

Now, fixing g, d, we consider all such data $(A/S, L)$ in the category of locally northerian schemes, it is natural to ask if there exists canonical choice of the isomorphism

$$\alpha_L : \det(\pi_* L)^{\otimes 2m} \cong (e^* \omega_A^\vee)^{\otimes md}$$

such that it is compatible with arbitrary base-change. That means if we are given a Cartesian diagram

$$\begin{tikzcd} A \times_S S' \arrow[r, "p_A"] \arrow[d, "\pi \times_S \mathrm{id}_{S'}"] & A \arrow[d, "\pi"] \\ S' \arrow[r, "f"] & S, \end{tikzcd}$$

then we always have $f^* \alpha_L = \alpha_{(p_A^* L)}$. Moret-Bailly shows in [MB, Chapitre VIII, Théorème 3.2] that this is true when d is invertible on S. This is so called the canonical key formula for projective abelian schemes, and it is denoted by $\mathrm{FCC}^{\mathrm{ab}}(\mathrm{Spec}\mathbb{Z}[1/d], g, d)$.

Question A: Is there a canonical key formula for projective abelian schemes without restriction on d, namely $\mathrm{FCC}^{\mathrm{ab}}(\mathrm{Spec}\mathbb{Z}, g, d)$?

Another direction is to look for the order of $\Delta(L)$ in the Picard group $\mathrm{Pic}(S)$. When S is a scheme which is quasi-projective over an affine noetherian scheme, Chai and Faltings prove the following result (cf. [FC, Theorem 5.1, p. 25]).

Theorem 1.1. *(Chai-Faltings) There is an isomorphism* $\det(\pi_* L)^{\otimes 8d^3} \cong (e^* \omega_A^\vee)^{\otimes 4d^4}$ *of line bundles on* S.

This is to say that $4d^3$ is an upper bound of the order $\Delta(L)$ in $\mathrm{Pic}(S)$. Later, Chai and Faltings' result was refined by Polishchuk in [Pol]. He has proved that there exists a constant $N(g)$, which depends only on the relative dimension g of A over S, killing $\Delta(L)$ in $\mathrm{Pic}(S)$. And he has also given various bounds for $N(g)$, which depend on d, on g and on the residue characteristics of S.

In a recent work [MR], Maillot and Rössler made a great progress in looking for the order of $\Delta(L)$. They prove the following results.

Theorem 1.2. *(Maillot-Rössler) (i). There is an isomorphism* $\Delta(L)^{\otimes 12} \cong \mathcal{O}_S$.

(ii). For every $g \geq 1$*, there exist data* $\pi : A \to S$ *and* L *as above such that* $\dim(A/S) = g$ *and such that* $\Delta(L)$ *is of order* 12 *in the Picard group of* S.

Now, suppose that the generic fibre of S is separated and smooth, then S can be viewed as an "arithmetic scheme" in the sense of Gillet-Soulé and $A(\mathbb{C})$ is a family of abelian varieties over $S(\mathbb{C})$. So it is an interesting problem that studying the trivialization of some power of

$\Delta(L)$ in an arithmetic sense according to the theory of Arakelov geometry. To be more precise, notice that given a Kähler fibration structure on $\pi_{\mathbb{C}}: A(\mathbb{C}) \to S(\mathbb{C})$, any hermitian metric on $L_{\mathbb{C}}$ induces a canonical metric on the determinant bundle $\det(\pi_* L)$ i.e. the Quillen metric on the determinant (cf. Section 4.1, below). Moreover, this Kähler fibration structure implies a hermitian metric on Ω_π, then we will get a hermitian metric on $\Delta(L)$. It will be denoted by $\Delta(\overline{L})$ the hermitian line bundle obtained in such a way.

Let us fix g, d and consider all data $(A/S, L)$ such that the generic fibre of S is separated and smooth, it is natural to ask the following.

Question B: Are there canonical metric on L and canonical Kähler fibration structure on $\pi_{\mathbb{C}}: A(\mathbb{C}) \to S(\mathbb{C})$ such that $\Delta(\overline{L})$ has a torsion class in the arithmetic Picard group $\widehat{\mathrm{Pic}}(S)$?

If the answer is YES, then we get an arithmetic canonical key formula for projective abelian schemes in the context of Arakelov geometry, which can be denoted by $\widehat{\mathrm{FCC}}^{\mathrm{ab}}(\mathrm{Spec}\mathbb{Z}, g, d)$.

The aim of this paper is to give positive answers to **Question A** and **Question B**, actually we provide a refinement of the canonical key formula for projective abelian schemes by indicating an explicit upper bound of the order of $\Delta(L)$. Our main theorem is the following.

Theorem 1.3. *Let A/S be a projective abelian scheme over a locally noetherian scheme S and let L be a symmetric, rigidified, relatively ample line bundle on A. Then*

(i). there is a trivialization of $\Delta(L)^{\otimes 12}$ which is canonical in the sense that it is compatible with arbitrary base-change;

(ii). if the generic fibre of S is separated and smooth, then there exist a positive integer m, a canonical metric on L and a canonical Kähler fibration structure on $\pi_{\mathbb{C}}: A(\mathbb{C}) \to S(\mathbb{C})$ such that there exists an isometry $\Delta(\overline{L})^{\otimes m} \cong \overline{\mathcal{O}}_S$ which is canonical in the sense of (i). Here the constant m only depends on g, d and is independent of L.

As a byproduct of (i) of this main theorem, the condition of quasi-projectivity on S in Theorem 1.2 can be removed. We also indicate that our main theorem can be viewed as a generalization of Moret-Bailly's work [MB1] where he considered the case $d = 1$.

The strategy we use to prove the first part of Theorem 1.3 is the representability of a moduli functor classifying projective abelian schemes with some additional structures, see Section 2 below for details. And the key input in the proof of (ii) of Theorem 1.3 is an arithmetic Adams-Riemann-Roch theorem in the context of Arakelov geometry, see [Roe] or below.

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2 Moduli functors classifying projective abelian schemes

Until the end of this paper, all schemes will be locally noetherian. Let S be a scheme, a group scheme $\pi : A \to S$ is called an abelian scheme if π is smooth and proper, and the geometric

fibres of π are connected. A basic fact (by the rigidity lemma) is that every abelian scheme is commutative.

2.1 Mumford's moduli functor $\mathcal{H}_{g,d,n}$

Definition 2.1. Let A/S be an abelian scheme with unit section $e : S \to A$.

(i). A line bundle L on A is said to be rigidified if e^*L is isomorphic to $\mathcal{O}_S$;

(ii). The relative Picard functor $\mathbf{Pic}(A/S)$ is the functor which sends an S-scheme T to the set of isomorphism classes of rigidified line bundles on $A \times_S T$ with respect to the unit section $e_T := e \times_S \mathrm{id}_T$;

(iii). The subfunctor $\mathbf{Pic}^0(A/S)$ of $\mathbf{Pic}(A/S)$ is the functor which sends an S-scheme T to the set of isomorphism classes of rigidified line bundles L on $A \times_S T$ with respect to e_T such that for $\forall t \in T$, $L \otimes k(t)$ is algebraically equivalent to zero on A_t.

We remark that if A is projective over S, then $\mathbf{Pic}(A/S)$ is represented by a separated S-scheme which is locally of finite type over S and $\mathbf{Pic}^0(A/S)$ is represented by a projective abelian scheme over S. In this case, we shall call $\mathbf{Pic}^0(A/S)$ the dual abelian scheme of A/S and denote it by $A^\vee/S$.

Now let L be a rigidified line bundle on a projective abelian scheme A/S with unit section e. Let $m : A \times_S A \to A$ be the group law, $p_1, p_2 : A \times_S A \to A$ be the first and the second projection respectively. Consider the line bundle

$$\widetilde{L} := m^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}$$

on $A \times_S A$. Regarding $A \times_S A$ as an abelian scheme over A with the unit section $e \times_S \mathrm{id}_A$, then $\widetilde{L}$ is rigidified and for any $a \in A$, $\widetilde{L}$ is algebraically equivalent to zero on the fibre A_a which is an abelian variety. So $\widetilde{L}$ induces an S-morphism from A to $A^\vee$, it is actually a group homomorphism. We denote this homomorphism by $\lambda(L)$.

Definition 2.2. Let $\pi : A \to S$ be a projective abelian scheme. A polarization of A is an S-homomorphism

$$\lambda : A \to A^\vee$$

such that, for any geometric point $\bar{s}$ of S, the induced morphism $\lambda_{\bar{s}}$ is of the form $\lambda(L_{\bar{s}})$ where $L_{\bar{s}}$ is an ample line bundle on $A_{\bar{s}}$.

If $\lambda : A \to A^\vee$ is a polarization, λ is finite and faithfully flat. The pull-back of the Poincaré line bundle along $(\mathrm{id}_A, \lambda) : A \to A \times_S A^\vee$ is a symmetric, rigidified and relatively ample line bundle $L^\Delta(\lambda)$ such that $\lambda(L^\Delta(\lambda)) = 2\lambda$ (cf. [FC, Chapter I, 1.6] and [MFK, Chapter 6, §2]).

Lemma 2.3. *Let L be a rigidified, relatively ample line bundle on a projective abelian scheme A/S. Then the line bundle $L^\Delta(\lambda(L))$ is canonically isomorphic to $L \otimes [-1]^*L$.*

Proof. By the construction of $\lambda(L)$, the pull-back of the Poincaré line bundle along $\mathrm{id}_A \times_S \lambda : A \times_S A \to A \times_S A^\vee$ is isomorphic to $m^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}$. Denote by $\Delta : A \to A \times_S A$ the diagonal morphism and by $[n] : A \to A$ the homomorphism of multiplication by n, then we have

$$\begin{aligned} L^\Delta(\lambda(L)) &\cong \Delta^*\big(m^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}\big) \\ &= [2]^* L \otimes L^{-2} \\ &\cong L \otimes [-1]^* L. \end{aligned}$$

The last isomorphism follows from the theorem of the cube. □

Definition 2.4. Let $\pi : A \to S$ be an abelian scheme with unit section e. Let n be a positive integer. Assume that A/S has relative dimension g and that the characteristics of the residue fields of all $s \in S$ do not divide n. Then if $n \geq 2$, a level-n-structure on A/S is a set of $2g$ sections $\sigma_1, \sigma_2, \ldots, \sigma_{2g}$ of π such that

(i). for all geometric points $\bar{s}$ of S, the images $\sigma_i(\bar{s})$ form a basis for the group of points of order n on the fibre $A_{\bar{s}}$;

(ii). $[n] \circ \sigma_i = e$ for $i = 1, 2, \ldots, 2g$.

It is convenient to call A/S by itself a level-1-structure.

Definition 2.5. Let g, d, n be three positive integers. The moduli functor $\mathcal{A}_{g,d,n}$ is the contravariant functor from the category of schemes to the category of sets which sends any scheme S to the set of isomorphism classes of the following data:

(i). a projective abelian scheme A over S of relative dimension g;

(ii). a polarization $\lambda : A \to A^\vee$ of degree d^2, i.e. $\lambda_*(\mathcal{O}_A)$ is locally free of rank d^2;

(iii). a level-n-structure of A over S.

We say that $(A/S, \lambda)$ is isomorphic to $(A'/S, \lambda')$ if there exists an S-isomorphism of abelian schemes $\gamma : A \to A'$ which induces an S-isomorphism of abelian schemes $\gamma^\vee : A'^\vee \to A^\vee$, such that $\lambda = \gamma^\vee \circ \lambda' \circ \gamma$. If $\mathcal{A}_{g,d,n}$ is represented by a scheme $A_{g,d,n}$, then $A_{g,d,n}$ will be called a fine moduli scheme.

Let $\pi : A \to S$ be a projective abelian scheme of relative dimension g, and let $\lambda : A \to A^\vee$ be a polarization of A/S of degree d^2. Then $\pi_*(L^\Delta(\lambda)^3)$ is a locally free sheaf on S of rank $6^g \cdot d$ (cf. [MFK, Prop. 6.13]). In this case, a linear rigidification of A/S associated to λ is an S-isomorphism $\mathbb{P}(\pi_*(L^\Delta(\lambda)^3)) \cong \mathbb{P}_S^{6^g \cdot d - 1}$.

Definition 2.6. Let g, d, n be three positive integers. The moduli functor $\mathcal{H}_{g,d,n}$ is the contravariant functor from the category of schemes to the category of sets which sends any scheme S to the set of isomorphism classes of the following data:

(i). a projective abelian scheme A over S of relative dimension g;

(ii). a polarization $\lambda : A \to A^\vee$ of degree d^2;

(iii). a level-n-structure of A over S;

(iv). a linear rigidification $\mathbb{P}(\pi_*(L^\Delta(\lambda)^3)) \cong \mathbb{P}_S^{6^g \cdot d-1}$.

Lemma 2.7. *Suppose that we have an S-isomorphism of polarized projective abelian schemes $\gamma : (A/S, \lambda) \cong (A'/S, \lambda')$, then $\gamma^* L^\Delta(\lambda')$ is canonically isomorphic to $L^\Delta(\lambda)$.*

Proof. We only need to show that $\gamma^* L^\Delta(\lambda') \cong L^\Delta(\gamma^\vee \circ \lambda' \circ \gamma)$ because $\lambda = \gamma^\vee \circ \lambda' \circ \gamma$. Let $\mathcal{P}'$ be the Poincaré line bundle on $A' \times_S A'^\vee$, then by definition we have

$$\begin{aligned}
\gamma^* L^\Delta(\lambda') =& \gamma^* \Delta_{A'}^* (\mathrm{id}_{A'} \times \lambda')^* (\mathcal{P}') \\
=& \Delta_A^* (\gamma \times \gamma)^* (\mathrm{id}_{A'} \times \lambda')^* (\mathcal{P}') \\
=& \Delta_A^* (\mathrm{id}_{A'} \times \lambda' \circ \gamma \times \gamma)^* (\mathcal{P}') \\
=& \Delta_A^* \big(\gamma \times (\lambda' \circ \gamma)\big)^* (\mathcal{P}')
\end{aligned}$$

On the other hand, we recall the definition of $\gamma^\vee$, it is the scheme morphism corresponding to the natural transformation $\alpha : \mathrm{Pic}^0(A'/S) \to \mathrm{Pic}^0(A/S)$. For any S-scheme T, $\alpha(T)$ sends the rigidified line bundles on $A' \times_S T$ to $A \times_S T$ by doing pull-back along $\gamma \times \mathrm{id}_T$. Hence $\gamma^\vee$ corresponds to the rigidified line bundle $(\gamma \times \mathrm{id}_{A'^\vee})^*(\mathcal{P}')$ on $A \times_S A'^\vee$. Let $\mathcal{P}$ be the Poincaré line bundle on $A \times_S A^\vee$, then by its universal property we have $(\gamma \times \mathrm{id}_{A'^\vee})^*(\mathcal{P}') \cong (\mathrm{id}_A \times \gamma^\vee)^*(\mathcal{P})$. Therefore

$$\begin{aligned}
L^\Delta(\gamma^\vee \circ \lambda' \circ \gamma) =& \Delta_A^* \big(\mathrm{id}_A \times (\gamma^\vee \circ \lambda' \circ \gamma)\big)^* (\mathcal{P}) \\
=& \Delta_A^* \big(\mathrm{id}_A \times (\lambda' \circ \gamma)\big)^* (\mathrm{id}_A \times \gamma^\vee)^* (\mathcal{P}) \\
\cong& \Delta_A^* \big(\mathrm{id}_A \times (\lambda' \circ \gamma)\big)^* (\gamma \times \mathrm{id}_{A'^\vee})^* (\mathcal{P}') \\
=& \Delta_A^* \big(\gamma \times (\lambda' \circ \gamma)\big)^* (\mathcal{P}')
\end{aligned}$$

So we are done. □

Theorem 2.8. *(Mumford) For any positive integers g, d, n, the moduli functor $\mathcal{H}_{g,d,n}$ is represented by a quasi-projective scheme $H_{g,d,n}$ over $\mathbb{Z}$.*

Proof. This is [MFK, Proposition 7.3]. □

2.2 The $\mathbf{PGL}_N$-structure on the universal abelian scheme of $\mathcal{H}_{g,d,n}$

Define N to be the integer $6^g \cdot d$, then the group scheme $\mathbf{PGL}_N$ has an action on the moduli functor $\mathcal{H}_{g,d,n}$ by transforming the linear rigidification. Hence by Yoneda lemma, $H_{g,d,n}$ admits a $\mathbf{PGL}_N$-action. Similarly, the universal abelian scheme $Z_{g,d,n}$ over $H_{g,d,n}$ also admits a $\mathbf{PGL}_N$-action because it represents the functor of linearly rigidified polarized projective abelian schemes with level-n-structure, and with one extra section. Although this is a well-known fact, we don't know a reference for its proof, so we include one.

Proposition 2.9. *Let $\mathcal{Z}_{g,d,n}$ be the moduli functor from the category of schemes to the category of sets which sends any scheme S to the set of isomorphism classes of the following data:*

(i). a projective abelian scheme A over S of relative dimension g;

(ii). a polarization $\lambda : A \to A^\vee$ of degree d^2;

(iii). a level-n-structure of A over S;

(iv). a linear rigidification $\mathbb{P}(\pi_*(L^\Delta(\lambda)^3)) \cong \mathbb{P}_S^{6^g \cdot d-1}$;

(v). a section $\epsilon : S \to A$.

Then $\mathcal{Z}_{g,d,n}$ is represented by the universal abelian scheme $Z_{g,d,n}$ of $\mathcal{H}_{g,d,n}$.

Proof. We first construct a natural transformation h from the functor $\mathrm{Hom}(\cdot, Z_{g,d,n})$ to the functor $\mathcal{Z}_{g,d,n}$, and then we prove that h is an isomorphism.

Consider the universal abelian scheme $\pi : Z_{g,d,n} \to H_{g,d,n}$ of the moduli functor $\mathcal{H}_{g,d,n}$, then the morphism π corresponds to the isomorphism class of the linearly rigidified polarized projective abelian scheme $p_2 : Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \to Z_{g,d,n}$ with level-n-structure. Denote by Δ the diagonal section $Z_{g,d,n} \to Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n}$.

For any scheme U and any morphism $f \in \mathrm{Hom}(U, Z_{g,d,n})$, we get a morphism $\pi \circ f \in \mathrm{Hom}(U, H_{g,d,n})$. We define $h(f)$ to be the isomorphism class of the linearly rigidified polarized projective abelian scheme $p_2 : Z_{g,d,n} \times_{H_{g,d,n}} U \to U$ which corresponds to the morphism $\pi \circ f$, with the section $\Delta \times \mathrm{id}_U$. This is reasonable because by construction $p_2 : Z_{g,d,n} \times_{H_{g,d,n}} U \to U$ is obtained from $Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \to Z_{g,d,n}$ by base-change along f. It is readily checked that h is actually a natural transformation from $\mathrm{Hom}(\cdot, Z_{g,d,n})$ to $\mathcal{Z}_{g,d,n}$.

For the injectivity of h, let f_1, f_2 be two morphisms from U to $Z_{g,d,n}$ such that $h(f_1) = h(f_2)$. Then by the definition of h, there exists an U-isomorphism from $Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \times_{f_1} U$ to $Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \times_{f_2} U$ compatible with all of their structures. This U-isomorphism induces an U-isomorphism δ from $Z_{g,d,n} \times_{\pi\circ f_1} U$ to $Z_{g,d,n} \times_{\pi\circ f_2} U$ compatible with all of their structures. Hence $\pi \circ f_1 = \pi \circ f_2$ and δ is actually the identity map because such U-automorphism of $Z_{g,d,n} \times_{H_{g,d,n}} U$ is already uniquely determined if we forget about the structure of one extra section (cf. the argument given before [MFK, Prop. 7.5]). This implies that the two sections ϵ_1 and ϵ_2 from U to $Z_{g,d,n} \times_{H_{g,d,n}} U$ induced by $\Delta \times_{f_1} \mathrm{id}_U$ and $\Delta \times_{f_2} \mathrm{id}_U$ respectively must be equal. Now, consider the following Cartesian diagram

$$\begin{tikzcd}
Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \times_{f_1} U \arrow[r, "p_3"] \arrow[d, "p_{12}"'] & U \arrow[d, "f_1"] \\
Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \arrow[r, "p_2"] & Z_{g,d,n},
\end{tikzcd}$$

we have $p_2 \circ p_{12} = f_1 \circ p_3$ which implies that $p_{12} = \Delta \circ f_1 \circ p_3$ and hence $p_{12} \circ \Delta \times_{f_1} \mathrm{id}_U = \Delta \circ f_1$. Since Δ is also a section of $p_1 : Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \to Z_{g,d,n}$, so we get $p_1 \circ p_{12} \circ \Delta \times_{f_1} \mathrm{id}_U = f_1$. Notice that $p_1 \circ p_{12} \circ \Delta \times_{f_1} \mathrm{id}_U$ is indeed $p_1 \circ \epsilon_1$, hence $f_1 = p_1 \circ \epsilon_1$. Similarly we have $f_2 = p_1 \circ \epsilon_2$, this finally implies that $f_1 = f_2$ because we already know that ϵ_1 is equal to ϵ_2.

For the surjectivity of h, let X/U be a linearly rigidified polarized projective abelian scheme with level-n-structure and with one extra section $\epsilon : U \to X$. Forgetting about the section $\epsilon : U \to X$, we get a morphism $l : U \to H_{g,d,n}$ since $H_{g,d,n}$ is a fine moduli scheme. Hence we may identify X/U with $p_2 : Z_{g,d,n} \times_{H_{g,d,n}} U \to U$ with all of their structures, the section $\epsilon : U \to X$ induces a section $U \to Z_{g,d,n} \times_{H_{g,d,n}} U$ which is still denoted by ϵ. Define $f = p_1 \circ \epsilon : U \to Z_{g,d,n}$, then it is clear that $\pi \circ f = l$. We want to show that $h(f)$ is exactly the isomorphism class of $p_2 : Z_{g,d,n} \times_{H_{g,d,n}} U \to U$ with the extra section ϵ. Denote by ϵ' the section of $p_2 : Z_{g,d,n} \times_{H_{g,d,n}} U \to U$ induced by the section $\Delta \times_f \mathrm{id}_U$, we only need to show that $\epsilon' = \epsilon$ since $\pi \circ f = l$ which implies the compatibilities of other structures. In fact, in the proof of the injectivity of h we have already known that $f = p_1 \circ \epsilon'$. But $f = p_1 \circ \epsilon$ follows from the definition, so the equality $\pi \circ f = l$ immediately implies that ϵ' and ϵ are both equal to the morphism $f \times \mathrm{id}_U$. So we are done. □

Via Yoneda lemma, the morphism $\pi : Z_{g,d,n} \to H_{g,d,n}$ corresponds to a morphism of functors from $\mathrm{Hom}(\cdot, Z_{g,d,n})$ to $\mathrm{Hom}(\cdot, H_{g,d,n})$. By the construction of h, this functor morphism is exactly the one from $\mathcal{Z}_{g,d,n}$ to $\mathcal{H}_{g,d,n}$ forgetting about the structure of one extra section. Therefore, the morphism $\pi : Z_{g,d,n} \to H_{g,d,n}$ is naturally $\mathbf{PGL}_N$-equivariant.

Next, we investigate possible $\mathbf{PGL}_N$-structures on quasi-coherent sheaves on $Z_{g,d,n}$ and $H_{g,d,n}$. Let G be a group scheme and let X be a scheme, recall that an action of G on X is a morphism $m_X : G \times X \to X$ which satisfies certain properties of compatibility (cf. [Kö]). Let F be a quasi-coherent sheaf on X, a G-action on F is an isomorphism of $\mathcal{O}_{G\times X}$-modules $\phi : m_X^* F \cong p_2^* F$ which satisfies the following cocycle condition on $G \times G \times X$:

$$(p_{23}^*\phi) \circ \big((\mathrm{id}_G \times m_X)^*\phi\big) = (m_G \times \mathrm{id}_X)^*\phi$$

where $m_G : G \times G \to G$ is the multiplication of G. Since we defined the $\mathbf{PGL}_N$-structures on $Z_{g,d,n}$ and on $H_{g,d,n}$ in a functorial way via Yoneda lemma, it is helpful to introduce the following functorial description of the group structures of quasi-coherent sheaves.

Theorem 2.10. *Let F be a quasi-coherent sheaf on X, then to give a G-structure on F is equivalent to give a family of $\mathcal{O}_A$-module isomorphisms $\{\phi^A_{g,x} : (gx)^*F \cong x^*F\}$ for each affine scheme A over X and for all A-valued points $g \in G(A), x \in X(A)$, such that*

(i). $\phi^A_{g,x} \circ \phi^A_{g',gx} = \phi^A_{g'g,x}$;

(ii). for any X-morphism of affine schemes $f : B \to A$, $\phi^A_{f\circ g, f\circ x}$ is the pull-back of $\phi^A_{g,x}$ along the morphism f;

Proof. Suppose that we are given a G-structure on F, which is an isomorphism $\phi : m_X^*F \cong p_2^*F$ satisfying certain property of associativity (a cocycle condition). Let A be an affine scheme over X, for any A-valued points $g \in G(A)$ and $x \in X(A)$ we have a morphism $u : A \to G \times X$. Then $gx \in X(A)$ is the morphism $m_X \circ u$. We define $\phi^A_{g,x} : (gx)^*F \cong x^*F$ to be the isomorphism which is the pull-back of ϕ along the morphism u. It is readily checked that this assignment satisfies the conditions (i) and (ii).

Conversely, suppose that we are given an assignment satisfying conditions (i) and (ii). We choose an open affine covering $\{A_i\}_{i\in I}$ of $G\times X$, the natural embedding $u_i : A_i \to G\times X$ gives A-valued points $g_i \in G(A_i)$ and $x_i \in X(A)$. Then $\phi^{A_i}_{g_i,x_i}$ provides an isomorphism $u_i^* m_X^* F \cong u_i^* p_2^* F$. These isomorphisms can be glued to get a global isomorphism $\phi : m_X^* F \cong p_2^* F$ because of (ii), and ϕ satisfies the cocycle condition because of (i). So we get a G-structures on F. $\square$

Now, we can describe the $\mathbf{PGL}_N$-structure on the line bundle $L^{\Delta}(\lambda)$ where λ is the universal polarization of the universal abelian scheme $Z_{g,d,n}$. Let A be an affine scheme over $Z_{g,d,n}$. Let $g \in \mathbf{PGL}_N(A)$ and $u \in Z_{g,d,n}(A)$ be two A-valued points. We shall denote gu by v for convenience. Then we need to give an isomorphism $\phi^A_{v,u} : v^* L^{\Delta}(\lambda) \cong u^* L^{\Delta}(\lambda)$. Consider the following Cartesian diagram:

$$\begin{tikzcd}
Z_{g,d,n} \arrow[d, "\pi"] & Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \arrow[l, "p_1"'] \arrow[d, "p_2"] & Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \times_v A \arrow[l, "p_{12}"'] \arrow[d, "p_3"] \\
H_{g,d,n} & Z_{g,d,n} \arrow[l, "\pi"'] & A. \arrow[l, "v"']
\end{tikzcd}$$

In the proof of Proposition 2.9, we showed that $v^* L^{\Delta}(\lambda)$ is equal to $(p_1 \circ p_{12} \circ \Delta \times_v \mathrm{id}_A)^* L^{\Delta}(\lambda)$ where Δ is the diagonal section from $Z_{g,d,n}$ to $Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n}$. Hence $v^* L^{\Delta}(\lambda)$ is canonically isomorphic to the pull-back of the line bundle $L^{\Delta}(\lambda_v)$ on $Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \times_v A$ along the section $\Delta \times_v \mathrm{id}_A$. Similarly, $u^* L^{\Delta}(\lambda)$ is canonically isomorphic to the pull-back of the line bundle $L^{\Delta}(\lambda_u)$ on $Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \times_u A$ along the section $\Delta \times_u \mathrm{id}_A$. But notice that the action of $\mathbf{PGL}_N(A)$ on $Z_{g,d,n}(A)$ is just the transformation of linear rigidifications which doesn't affect the structures of projective abelian scheme, of polarization, of level-n-structure and of the extra one section. So there exists another projective abelian scheme X over A with polarization λ_X, level-n-structure and one extra section ϵ_X such that there exist unique A-isomorphisms $\eta_v : X \to Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \times_v A$ and $\eta_u : X \to Z_{g,d,n} \times_{H_{g,d,n}} Z_{g,d,n} \times_u A$ which are compatible with all of their structures (except the structure of linear rigidifications!). By Lemma 2.7, we have canonical isomorphisms $\eta_v^* L^{\Delta}(\lambda_v) \cong L^{\Delta}(\lambda_X) \cong \eta_u^* L^{\Delta}(\lambda_u)$. Pulling back these isomorphisms along the section ϵ_X, we finally get an isomorphism $\phi^A_{v,u} : v^* L^{\Delta}(\lambda) \cong u^* L^{\Delta}(\lambda)$. It is easily seen that this isomorphism $\phi^A_{v,u}$ is independent of the choice of the representative X/A. Let $g' \in \mathbf{PGL}_N(A)$ be another A-valued point. Denote $g'v$ by w, we have to show that $\phi^A_{v,u} \circ \phi^A_{w,v} = \phi^A_{w,u}$. But this simply follows from the construction: $\eta_w^* L^{\Delta}(\lambda_w) \cong L^{\Delta}(\lambda_X) \cong \eta_v^* L^{\Delta}(\lambda_v) = \eta_v^* L^{\Delta}(\lambda_v) \cong L^{\Delta}(\lambda_X) \cong \eta_u^* L^{\Delta}(\lambda_u)$ is equal to $\eta_w^* L^{\Delta}(\lambda_w) \cong L^{\Delta}(\lambda_X) \cong \eta_u^* L^{\Delta}(\lambda_u)$. At last, $\phi^A_{v,u}$ is clearly functorial so that we get a $\mathbf{PGL}_N$-action on the geometric functor $\theta(L^{\Delta}(\lambda))$ and hence a $\mathbf{PGL}_N$-structure on $L^{\Delta}(\lambda)$. The $\mathbf{PGL}_N$-structure constructed like this way is called the canonical $\mathbf{PGL}_N$-structure. If a quasi-coherent sheaf on $Z_{g,d,n}$ comes from the data of structures in the definition of the representable functor $\mathcal{Z}_{g,d,n}$, then it is compatible with arbitrary base-change and we call it a universal quasi-coherent sheaf. For universal quasi-coherent sheaves on $Z_{g,d,n}$, it is always possible to construct canonical $\mathbf{PGL}_N$-structures. For instance, we may define the canonical $\mathbf{PGL}_N$-structure on the canonical sheaf $\omega_{Z_{g,d,n}/H_{g,d,n}}$.

Over $H_{g,d,n}$, the line bundle $\Delta(L^{\Delta}(\lambda))$ admits the canonical $\mathbf{PGL}_N$-structure because it

is a universal bundle. And the structure sheaf $\mathcal{O}_{H_{g,d,n}}$ admits a natural $\mathbf{PGL}_N$-structure because $H_{g,d,n}$ is $\mathbf{PGL}_N$-equivariant. Notice that the canonical $\mathbf{PGL}_N$-structures on $L^\Delta(\lambda)$ and on $\omega_{Z_{g,d,n}/H_{g,d,n}}$ induce a $\mathbf{PGL}_N$-structure on $\Delta(L^\Delta(\lambda))$ since $\pi : Z_{g,d,n} \to H_{g,d,n}$ is $\mathbf{PGL}_N$-equivariant. It can be checked that this is exactly the canonical $\mathbf{PGL}_N$-structure on $\Delta(L^\Delta(\lambda))$.

For any scheme S and for every positive integer k, we shall denote by $S[1/k]$ the scheme $S \times_{\mathbb{Z}} \{\mathrm{Spec}(\mathbb{Z}) - \bigcup_{p|k}(p)\}$ obtained by removing the fibres over p which divides k. To end this subsection, we summarize some facts about $H_{g,d,n}$.

Theorem 2.11. *Let g, d, n be any positive integers.*

(i). If $n > 6^g \cdot d \cdot \sqrt{g!}$, then the fine moduli scheme $A_{g,d,n}$ exists and $H_{g,d,n}$ is a $\mathbf{PGL}_N$-torsor over $A_{g,d,n}$. In this case, $A_{g,d,n}$ is faithfully flat over $\mathrm{Spec}\mathbb{Z}[1/n]$ *and is smooth over* $\mathrm{Spec}\mathbb{Z}[1/nd]$.

(ii). $H_{g,d,n}$ is faithfully flat over $\mathrm{Spec}\mathbb{Z}[1/n]$ *and is smooth over* $\mathbb{Q}$.

Proof. The statements in (i) are the contents of [MFK, Prop. 7.6, Thm. 7.9] and [Chai, Thm 1.4 (a)]. We prove (ii). Let k be a positive integer, suppose that $\sigma_1, \ldots, \sigma_{2g}$ is a level-nk-structure on a projective abelian scheme A/S. Then $k\sigma_1, \ldots, k\sigma_{2g}$ is a level-n-structure on A/S. This defines a morphism $p_n^{(k)}$ from $\mathcal{H}_{g,d,nk}$ to $\mathcal{H}_{g,d,n}$, and hence from $H_{g,d,nk}$ to $H_{g,d,n}$. Moreover let Γ_n be the group $\mathbf{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$, then Γ_n has a natural action on $\mathcal{H}_{g,d,n}$ and hence on $H_{g,d,n}$. Indeed, for any element $T = (a_{i,j}) \in \Gamma_n$ and any level-n-structure $\sigma_1, \ldots, \sigma_{2g}$ on A/S, the set of sections $\sum_{j=1}^{2g} a_{1,j}\sigma_j, \ldots, \sum_{j=1}^{2g} a_{2g,j}\sigma_j$ is also a level-n-structure. There is a canonical morphism from Γ_{nk} to Γ_n, we denote its kernel by $\Gamma_n^{(k)}$. In [MFK, Lemma 7.11], Mumford proved that $p_n^{(k)} : H_{g,d,nk} \to H_{g,d,n}[1/k]$ is a $\Gamma_n^{(k)}$-torsor and $p_n^{(k)}$ is actually a finite étale morphism.

Now, let g, d, n be any positive integers. According to (i), we may take an integer k big enough so that $A_{g,d,nk}$ exists and $H_{g,d,nk}$ is a $\mathbf{PGL}_N$-torsor over $A_{g,d,nk}$. So $H_{g,d,nk}$ is faithfully flat over $\mathrm{Spec}\mathbb{Z}[1/nk]$. Together with the faithfully flatness of $p_n^{(k)}$, we know that $H_{g,d,n}[1/k]$ is faithfully flat over $\mathrm{Spec}\mathbb{Z}[1/nk]$. Replacing k by another integer big enough which is prime to k, we finally obtain that $H_{g,d,n}$ is faithfully flat over $\mathrm{Spec}\mathbb{Z}[1/n]$. For the smoothness, firstly note that $H_{g,d,nk} \times_{\mathbb{Z}} \mathbb{Q}$ is smooth over $A_{g,d,nk} \times_{\mathbb{Z}} \mathbb{Q}$ since $\mathbf{PGL}_N$ is smooth over $\mathbb{Q}$, hence the generic fibre of $H_{g,d,nk}$ is smooth by (i) i.e. the generic fibre of $H_{g,d,nk}$ is regular. Again by the faithfully flatness of $p_n^{(k)}$, we know that the generic fibre of $H_{g,d,n}$ is regular. Therefore $H_{g,d,n}$ is smooth over $\mathbb{Q}$. $\square$

2.3 A variant of Mumford's moduli functor $\widetilde{\mathcal{H}}_{g,d,n}$

In this subsection, we shall introduce a variant of Mumford's moduli functor $\widetilde{\mathcal{H}}_{g,d,n}$ which classifies linearly rigidified projective abelian schemes with level-n-structure, and with a symmetric, rigidified ample line bundle.

Definition 2.12. Let g, d, n be three positive integers. The moduli functor $\widetilde{\mathcal{H}}_{g,d,n}$ is the contravariant functor from the category of schemes to the category of sets which sends any scheme S to the set of isomorphism classes of the following data:

(i). a projective abelian scheme $\pi : A \to S$ of relative dimension g, with unit section e;

(ii). a symmetric, rigidified and relatively ample line bundle L on A such that the rank of the vector bundle $\pi_*(L)$ is d;

(iii). a rigidification $e^*L \cong \mathcal{O}_S$;

(iv). a level-n-structure of A over S;

(v). a linear rigidification $\mathbb{P}(\pi_*(L^6)) \cong \mathbb{P}_S^{6^g \cdot d-1}$.

Lemma 2.13. *Let $(A/S, L)$ and $(A'/S, L')$ be two projective abelian schemes over S equipped with symmetric, rigidified ample line bundles. Suppose that there exists an S-isomorphism $\gamma : A \to A'$ which induces an S-isomorphism $\gamma^\vee : A'^\vee \to A^\vee$ such that $L \cong \gamma^* L'$. Then $\lambda(L)$ is equal to $\gamma^\vee \circ \lambda(L') \circ \gamma$.*

Proof. We only need to show that $\lambda(L') \circ \gamma = \gamma^{\vee -1} \circ \lambda(\gamma^* L')$ because L is isomorphic to $\gamma^* L'$. Consider the following diagram at first:

$$A \xrightarrow{\gamma} A' \xrightarrow{\lambda(L')} A'^\vee.$$

By the universal property of $A'^\vee$, the composition $\lambda(L') \circ \gamma$ corresponds to the rigidified line bundle $(\mathrm{id}_{A'} \times \gamma)^*(m_{A'}^* L' \otimes p_{1A'}^* L'^{-1} \otimes p_{2A'}^* L'^{-1})$ on $A' \times_S A$.

On the other hand, consider the following diagram:

$$A \xrightarrow{\lambda(\gamma^* L')} A^\vee \xrightarrow{\gamma^{\vee -1}} A'^\vee.$$

Recall that for any S-scheme T, the morphism $\gamma^{\vee -1}$ sends the elements of the relative Picard functor $\mathrm{Pic}^0(A/S)(T)$ to $\mathrm{Pic}^0(A'/S)(T)$ by doing pull-back along $(\gamma^{-1} \times \mathrm{id}_T)$. Then by the definition of $\lambda(\gamma^* L')$ we know that the composition $\gamma^{\vee -1} \circ \lambda(\gamma^* L')$ corresponds to the rigidified line bundle

$$\begin{aligned}
&(\gamma^{-1} \times \mathrm{id}_A)^*(m_A^* \gamma^* L' \otimes p_{1A}^* \gamma^* L'^{-1} \otimes p_{2A}^* \gamma^* L'^{-1}) \\
=&(\gamma^{-1} \times \mathrm{id}_A)^*(\gamma \times \gamma)^*(m_{A'}^* L' \otimes p_{1A'}^* L'^{-1} \otimes p_{2A'}^* L'^{-1}) \\
=&(\gamma \times \gamma \circ \gamma^{-1} \times \mathrm{id}_A)^*(m_{A'}^* L' \otimes p_{1A'}^* L'^{-1} \otimes p_{2A'}^* L'^{-1}) \\
=&(\mathrm{id}_{A'} \times \gamma)^*(m_{A'}^* L' \otimes p_{1A'}^* L'^{-1} \otimes p_{2A'}^* L'^{-1})
\end{aligned}$$

on $A' \times_S A$. So we are done. □

Let $\pi : A \to S$ be a projective abelian scheme, and let L be a symmetric, rigidified ample line bundle on A. Then L induces a polarization $\lambda(L) : A \to A^\vee$ such that $L^\Delta(\lambda(L))$ is canonically

isomorphic to L^2 (cf. Lemma 2.3). Notice that the square of the rank of $\pi_*(L)$ is equal to the degree of the polarization $\lambda(L)$, then according to Lemma 2.13 and Lemma 2.7, we have a well-defined natural transformation α from $\widetilde{\mathcal{H}}_{g,d,n}$ to $\mathcal{H}_{g,d,n}$. The following property of rigidified line bundle is very important for our later arguments.

Remark 2.14. Let L_1, L_2 be two rigidified line bundles on an abelian scheme $\pi : A \to S$ which are isomorphic to each other. Then there is only one isomorphism between L_1 and L_2 which is compatible with the rigidifications.

Remark 2.15. Let L be a rigidified line bundle on A and let ρ_1, ρ_2 be two rigidifications $e^*L \to \mathcal{O}_S$. By Remark 2.14, there exists a unique automorphism l of L which is compatible with ρ_1 and ρ_2. If L is moreover relatively ample and is equipped with some linear rigidification $\mathbb{P}(\pi_*(L^6)) \cong \mathbb{P}_S^{6^g \cdot d-1}$, then the automorphism l respects the linear rigidification. This result follows from the construction of the automorphism l, the projection formula and the fact that $\mathbf{PGL}_N$ is isomorphic to $\mathbf{GL}_{N+1}/\mathbf{G}_m$.

By Theorem 2.7, $\mathcal{H}_{g,d,n}$ is represented by a quasi-projective scheme $H_{g,d,n}$ over $\mathbb{Z}$. Consider the category $\mathcal{C}$ consisting of schemes over $H_{g,d,n}$, endowed with the fppf topology. Then $\widetilde{\mathcal{H}}_{g,d,n}$ induces a moduli functor $\widetilde{\mathcal{H}}'_{g,d,n}$ over $\mathcal{C}$ which sends an object $f : U \to H_{g,d,n}$ to the set of elements of $\widetilde{\mathcal{H}}_{g,d,n}(U)$ which have the same image under α in $\mathcal{H}_{g,d,n}(U)$ corresponding to f.

Proposition 2.16. *The moduli functor $\widetilde{\mathcal{H}}'_{g,d,n}$ is a sheaf over the site $\mathcal{C}$.*

Proof. Let $U \to H_{g,d,n}$ be an object in $\mathcal{C}$ and let $\{U_i \to H_{g,d,n}\}_{i\in I}$ be an fppf covering of U. We need to show that the following diagram

$$\widetilde{\mathcal{H}}'_{g,d,n}(U \to H_{g,d,n}) \longrightarrow \prod_i \widetilde{\mathcal{H}}'_{g,d,n}(U_i \to H_{g,d,n}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{i,j} \widetilde{\mathcal{H}}'_{g,d,n}(U_i \times_U U_j \to H_{g,d,n})$$

is an equalizer.

Firstly, let a, b be two elements in $\widetilde{\mathcal{H}}'_{g,d,n}(U \to H_{g,d,n})$ such that $a\mid_{U_i} = b\mid_{U_i}$ for any $i \in I$. Since $\alpha(a) = \alpha(b)$, we may assume that a, b are represented by $(A/U, L_1)$ and $(A/U, L_2)$ with the same level-n-structure and the same linear rigidification. Then $a\mid_{U_i} = b\mid_{U_i}$ means that there exists an U-automorphism η_i of $A \times_U U_i$ such that $L_1\mid_{U_i} \cong \eta_i^* L_2\mid_{U_i}$. Moreover, this U-automorphism η_i respects all of the structures appearing in the definition of $\mathcal{H}_{g,d,n}$, we again use Mumford's argument given before [MFK, Prop. 7.5] to conclude that such U-automorphism is unique. Hence η_i is the identity map so that $L_1\mid_{U_i} \cong L_2\mid_{U_i}$ for every $i \in I$. We take into account the rigidifications of L_1 and L_2, by Remark 2.14, the isomorphism $L_1\mid_{U_i} \cong L_2\mid_{U_i}$ is unique because it is compatible with the rigidifications. Therefore this family of isomorphisms are compatible on $U_i \times_U U_j$ so that $L_1 \cong L_2$ because the fibred category of quasi-coherent sheaves over the category of schemes is a stack with respect to the fppf topology (cf. [Vi, Thm. 4.23]). The resulting isomorphism $L_1 \cong L_2$ is compatible with the rigidifications since its restriction to every U_i is so. We finally have $a = b$.

Secondly, let $\prod_i a_i$ be an element in $\prod_i \widetilde{\mathcal{H}}'_{g,d,n}(U_i \to H_{g,d,n})$ with $a_i \in \widetilde{\mathcal{H}}'_{g,d,n}(U_i \to H_{g,d,n})$ such that $p_1^*(\prod_i a_i) = p_2^*(\prod_i a_i)$. So we have $p_1^*(\prod_i \alpha(a_i)) = p_2^*(\prod_i \alpha(a_i))$ in $\prod_{i,j} \mathcal{H}_{g,d,n}(U_i \times_U U_j)$. Then by the fact that every representable functor is a sheaf with respect to the fppf topology, we have an element $t \in \mathcal{H}_{g,d,n}(U)$ such that $t\mid_{U_i} = \alpha(a_i)$. Therefore we may assume that there exists a linearly rigidified polarized projective abelian scheme $(A/U, \lambda)$ with a level-n-structure and every a_i is equal to $(A \times_U U_i/U_i, L_i)$ with the induced level-n-structure such that $\lambda(L_i) = \lambda\mid_{U_i}$. Now, we take the rigidification of L_i into account and use the descent theory for quasi-projective morphisms via relatively ample line bundles. Precisely, notice that $p_1^*(\prod_i a_i) = p_2^*(\prod_i a_i)$ and the isomorphism between L_j and L_i on $A \times_U U_i \times_U U_j$ is required to be compatible with the rigidifications, we may glue all $(A \times_U U_i/U_i, L_i)$ to get a scheme P which is quasi-projective over U and a relatively ample line bundle L_P on P such that there exists a family of U-isomorphisms $\beta_i : (P, L_P)\mid_{U_i} \cong (A \times_U U_i/U_i, L_i)$ satisfying certain condition of compatibilities. The structure morphism $P \to U$ is moreover flat and proper since it is so after base-change along a faithfully flat morphism. Next, we may glue all isomorphisms β_i to get a global U-morphism β from P to A, this follows from the fact that the fibred category associated to a stable class of morphisms (e.g. flat morphisms) over the category of schemes is a prestack with respect to the fppf topology (cf. [Vi, Prop. 4.31]). The morphism β is actually an isomorphism because it becomes an isomorphism after base-change along a faithfully flat morphism. We transfer L_P via β to get a relatively ample line bundle L on A, then $L\mid_{U_i}$ is isomorphic to L_i for any $i \in I$. Finally, we still have to show that L is symmetric and rigidified. But, this can be easily seen from Remark 2.14 and the fact that the fibred category of quasi-coherent sheaves over the category of schemes is a stack with respect to the fppf topology (cf. [Vi, Thm. 4.23]), because every L_i is symmetric and rigidified by definition. So we are done. □

Now, let $\mathcal{G}$ be the group functor of $\mathcal{C}$ which sends an object $U \to H_{g,d,n}$ to the group of 2-torsion points of the dual of $Z_{g,d,n} \times_{H_{g,d,n}} U$. Then $\mathcal{G}$ is clearly represented by the subscheme of 2-torsion points of the dual of $Z_{g,d,n}$. We denote this scheme by G, it is finite flat over $H_{g,d,n}$ and is étale over $H_{g,d,n}[1/2]$.

To end this subsection, we mention that G has a natural action on $\widetilde{\mathcal{H}}'_{g,d,n}$. Let A/S be a projective abelian scheme and let L be a symmetric, rigidified ample line bundle on A. Then for any rigidified 2-torsion line bundle E (which is automatically symmetric), $E \otimes L$ and L induce the same polarization. In fact, $L^\Delta(\lambda(E \otimes L)) = L^\Delta(\lambda(L)) = L^2$ and hence $2\lambda(E \otimes L) = 2\lambda(L)$ which implies that $\lambda(E \otimes L) = \lambda(L)$. So we may define the action of G on $\widetilde{\mathcal{H}}'_{g,d,n}$ by twisting the relatively ample line bundle by a rigidified 2-torsion line bundle. By Remark 2.15 this action is well-defined, namely it is independent of the choice of the explicit rigidification of a rigidified 2-torsion line bundle. This G-action will play a crucial role in the study of the representability of $\widetilde{\mathcal{H}}_{g,d,n}$.

2.4 Representability of $\widetilde{\mathcal{H}}_{g,d,n}$

In this subsection, we shall investigate the representability of the functor $\widetilde{\mathcal{H}}_{g,d,n}$. It is clear that $\widetilde{\mathcal{H}}_{g,d,n}$ is representable if and only if $\widetilde{\mathcal{H}}'_{g,d,n}$ is representable. So we may concentrate on the

representability of $\widetilde{\mathcal{H}}'_{g,d,n}$. Our first result is the following.

Lemma 2.17. *Let λ be the universal polarization of the universal abelian scheme $Z_{g,d,n}$ over $H_{g,d,n}$. Then there exists an fppf covering $\{U_i \to H_{g,d,n}\}_{i\in I}$ of $H_{g,d,n}$ such that for every $i \in I$ there exists a symmetric, rigidified ample line bundle L_i on $Z_{g,d,n} \times_{H_{g,d,n}} U_i$ as a square root of $L^{\Delta}(\lambda)_{U_i}$.*

Proof. The proof given here is due to Tong (private communication between Tong and the author). In the following, we shall use the notation A/S instead of $Z_{g,d,n}/H_{g,d,n}$. We first prove that there exists an fppf covering of S such that locally on the fppf topology, $L^{\Delta}(\lambda)$ is isomorphic to $[2]^*M$ for some rigidified line bundle M. It is sufficient to show that locally on fppf topology, $L^{\Delta}(\lambda)$ admits an action of $\ker([2])$ which is compatible with the action of $\ker([2])$ on A given by translations.

In fact, denote by λ' the polarization defined by $L^{\Delta}(\lambda)$, then $\lambda' = 2\lambda$ so that we have the inclusion $\ker([2]) \subset \ker(\lambda')$. This implies that for any point $a \in \ker([2])$, $t_a^*L^{\Delta}(\lambda) \simeq L^{\Delta}(\lambda)$ where $t_a : A \to A$ is the translation map with respect to a. Consider the sheaf $K(L^{\Delta}(\lambda)) = \{(a,\alpha) \mid a \in \ker([2]), \alpha : t_a^*L^{\Delta}(\lambda) \simeq L^{\Delta}(\lambda)\}$, it fits a short exact sequence

$$0 \to \mathbf{G}_m \to K(L^{\Delta}(\lambda)) \to \ker([2]) \to 0.$$

Since the fppf sheaf $\mathcal{E}xt^1_S(\ker([2]), \mathbf{G}_m)$ is trivial (cf. [Ray, Lemme 6.2.2]), we may replace S by an fppf localization and suppose that the exact sequence given above is split. Hence there exists a section $\theta : \ker([2]) \to K(L^{\Delta}(\lambda))$ of group schemes over S. This section gives an action of $\ker([2])$ on $L^{\Delta}(\lambda)$ which is compatible with the action of $\ker([2])$ on A given by translations.

Now, for any $s \in S$, we may assume that there exists an fppf neighborhood V of s such that over A_V, $[2]^*M_V \simeq L^{\Delta}(\lambda)_{A_V}$ for some rigidified line bundle M_V. Since $[2]^*M_V$ is algebraically equivalent to $M_V^{\otimes 4}$, we have $N := L^{\Delta}(\lambda)_{A_V} \otimes M_V^{-4} \in A_V^{\vee}$. That means N induces a section $\eta : V \to \mathrm{Pic}^0(A_V/V)$. Consider the following Cartesian diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \mathrm{Pic}^0(A_V/V) \arrow[d, "{[2]}"] \\ V \arrow[r, "\eta"] & \mathrm{Pic}^0(A_V/V), \end{tikzcd}$$

here $[2]$ is faithfully flat. Therefore over A_W, N has a rigidified square root and hence $L^{\Delta}(\lambda)$ has a square root which is rigidified by construction. We denote this square root by L_W.

Now let Q be the rigidified line bundle $L_W \otimes [-1]^*L_W^{\vee}$. This line bundle Q is a 2-torsion since $L_W^{\otimes 2} = L^{\Delta}(\lambda)_{A_W}$ which is symmetric. Therefore Q induces a section $\beta : W \to \mathrm{Pic}^0(A_W/W)$. Consider the following Cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathrm{Pic}^0(A_W/W) \arrow[d, "{[2]}"] \\ W \arrow[r, "\beta"] & \mathrm{Pic}^0(A_W/W), \end{tikzcd}$$

here [2] is again faithfully flat. So over A_P, Q has a rigidified square root E. Denote by L_P the tensor product $L_W \otimes E$, then L_P is rigidified, symmetric and L_P induces the same polarization λ_{A_P} as L_W because E is a torsion bundle. This actually implies that E is a 2-torsion since $L_P^{\otimes 2} = L^{\Delta}(\lambda)_{A_P} = L_W^{\otimes 2}$ over A_P. So over A_P, Q is the trivial bundle and L_W is symmetric. Finally, to get the fppf covering $\{U_i \to S\}_{i \in I}$ we just take such fppf neighborhoods P around all the points of S. Notice that I can be chosen to be of finite number, because S is quasi-compact and flat morphisms are open if they are of finite type. $\square$

Corollary 2.18. *$\widetilde{\mathcal{H}}'_{g,d,n}$ is a G-torsor sheaf over the site $\mathcal{C}$.*

Proof. It is easily seen that the action of G on $\widetilde{\mathcal{H}}'_{g,d,n}$ is free and transitive. Then $\widetilde{\mathcal{H}}'_{g,d,n}$ is a G-torsor sheaf if and only if there exists an fppf covering $\{U_i \to H_{g,d,n}\}_{i \in I}$ of $H_{g,d,n}$ such that for every $i \in I$ the set $\widetilde{\mathcal{H}}'_{g,d,n}(U_i \to H_{g,d,n})$ is non-empty, because G is an abelian group. This fact follows from Lemma 2.17, so we are done. $\square$

Theorem 2.19. *The moduli functor $\widetilde{\mathcal{H}}'_{g,d,n}$ is representable.*

Proof. G is finite and hence affine over $H_{g,d,n}$, so the representability of $\widetilde{\mathcal{H}}'_{g,d,n}$ follows from Corollary 2.18 and [Mil, Theorem III.4.3 (a)]. $\square$

From Theorem 2.19 we know that the moduli functor $\widetilde{\mathcal{H}}_{g,d,n}$ is represented by a scheme $\widetilde{H}_{g,d,n}$ over $H_{g,d,n}$. The following proposition summarizes some properties of $\widetilde{H}_{g,d,n}$.

Proposition 2.20. *$\widetilde{H}_{g,d,n}$ is flat and quasi-projective over $\mathbb{Z}$, and it is smooth over $\mathbb{Q}$.*

Proof. G is finite flat over $H_{g,d,n}$ and $G[1/2]$ is étale over $H_{g,d,n}[1/2]$, so $\widetilde{H}_{g,d,n}$ is finite flat over $H_{g,d,n}$ and $\widetilde{H}_{g,d,n}[1/2]$ is étale over $H_{g,d,n}[1/2]$. This follows from [Mil, Prop. III.4.2]. Then the statement can be deduced from Theorem 2.11. $\square$

To end this subsection, we mention that the scheme $\widetilde{H}_{g,d,n}$ and the universal abelian scheme $\widetilde{Z}_{g,d,n}$ over $\widetilde{H}_{g,d,n}$ admit natural $\mathbf{PGL}_N$-actions such that the structure morphism $\pi : \widetilde{Z}_{g,d,n} \to \widetilde{H}_{g,d,n}$ is $\mathbf{PGL}_N$-equivariant. Moreover, the universal line bundle L on $\widetilde{Z}_{g,d,n}$ can be equipped with the canonical $\mathbf{PGL}_N$-structure. Similarly, by changing the rigidification, $\widetilde{H}_{g,d,n}$ and $\widetilde{Z}_{g,d,n}$ are $\mathbf{G}_m$-equivariant schemes. But by Remark 2.15, these $\mathbf{G}_m$-actions are both trivial. The universal line bundle L on $\widetilde{Z}_{g,d,n}$ can also be equipped with the canonical $\mathbf{G}_m$-structure which is not the trivial one.

3 Construction of the canonical trivialization of $\Delta(L)^{\otimes 12}$

Let $\pi : A \to S$ be a projective abelian scheme of relative dimension g with a symmetric, rigidified ample line bundle L such that the rank of $\pi_* L$ is equal to d. Here S is not necessarily

quasi-projective over an affine scheme. In this subsection, we shall construct an isomorphism $\Delta(L)^{\otimes 12} \cong \mathcal{O}_S$ which is canonical in the sense that it is compatible with arbitrary base-change.

Suppose that $\{U_i\}$ is an open covering of S such that the restriction of A/S to every U_i admits a linear rigidification. It is clear that such open covering always exists and moreover we may assume that all U_i are affine. We choose a linear rigidification for A_{U_i}/U_i, then there exists a unique morphism $f : U_i \to \widetilde{H}_{g,d,1}$ such that A_{U_i}/U_i is isomorphic to $\widetilde{Z}_{g,d,1} \times_f U_i/U_i$ with the structure of rigidified line bundle and the structure of linear rigidification. Write L_Z for the universal rigidified line bundle on $\widetilde{Z}_{g,d,1}$. By Proposition 2.20 we know that $\widetilde{H}_{g,d,1}$ is quasi-projective over $\mathbb{Z}$, then we may use the theorem of Maillot and Rössler (cf. Theorem 1.2) to conclude that the order of $\Delta(L_Z)$ in $\mathrm{Pic}(\widetilde{H}_{g,d,1})$ is a divisor of 12. We choose an arbitrary trivialization $\eta : \Delta(L_Z)^{\otimes 12} \cong \mathcal{O}_{\widetilde{H}_{g,d,1}}$, then it becomes universal. Pulling back η to U_i along f, we get an isomorphism η_i between $\Delta(L_i)^{\otimes 12}$ and $\mathcal{O}_{U_i}$. We want to show that η_i is independent of the choice of the linear rigidification for A_{U_i}/U_i. Note that A_{U_i}/U_i can be chosen as a representative to define the canonical $\mathbf{PGL}_N$-structure on $\Delta(L_Z)$, then the statement that η_i is independent of the choice of the linear rigidification is equivalent to the statement that the isomorphism $\eta : \Delta(L_Z)^{\otimes 12} \cong \mathcal{O}_{\widetilde{H}_{g,d,1}}$ is $\mathbf{PGL}_N$-equivariant. But η is automatically $\mathbf{PGL}_N$-equivariant because of the following lemma.

Lemma 3.1. *Every line bundle L on $\widetilde{H}_{g,d,1}$ admits at most one $\mathbf{PGL}_N$-structure.*

Proof. Let $m : \mathbf{PGL}_N \times \widetilde{H}_{g,d,1} \to \widetilde{H}_{g,d,1}$ be the $\mathbf{PGL}_N$-action on $\widetilde{H}_{g,d,1}$. A $\mathbf{PGL}_N$-structure on L is an isomorphism $\gamma : m^*L \cong p_2^*L$ which satisfies certain property of associativity. We first prove that two $\mathbf{PGL}_N$-structures γ_1 and γ_2 of L are equal if they are equal on the generic fibre. Actually, by Proposition 2.20 we know that $\widetilde{H}_{g,d,1}$ is flat over $\mathbb{Z}$ hence $\mathcal{O}_{H_{g,d,1}}$ is a flat $\mathbb{Z}$-module. Therefore the restriction map from $\mathcal{O}_{\mathbf{PGL}_N \times \widetilde{H}_{g,d,1}}$ to $\mathcal{O}_{\mathbf{PGL}_N \times \widetilde{H}_{g,d,1}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective. But p_2^*L is a flat $\mathcal{O}_{\mathbf{PGL}_N \times \widetilde{H}_{g,d,1}}$-module, so the restriction map $p_2^*L \to p_2^*L_{\mathbb{Q}}$ is injective. Hence if γ_1 and γ_2 are equal over generic fibre, then they must be equal globally. By Proposition 2.20, the generic fibre of $\widetilde{H}_{g,d,1}$ is smooth which implies that $(\widetilde{H}_{g,d,1})_{\mathbb{Q}}$ is geometrically reduced. Moreover, $(\mathbf{PGL}_N)_{\mathbb{Q}}$ is connected and there are no non-trivial characters $\mathbf{PGL}_N \to \mathbf{G}_m$. Thus we can use [MFK, Prop. 1.4] to conclude that $L_{\mathbb{Q}}$ admits only one $\mathbf{PGL}_N$-structure. So we are done. $\square$

Now we have known that the isomorphism η_i is really independent of the choice of the linear rigidification, hence η_i and η_j are equal on $U_i \times_S U_j$ so that we may glue all $\{\eta_i\}$ to get a global isomorphism $\alpha : \Delta(L)^{\otimes 12} \cong \mathcal{O}_S$. Clearly, the fact that η_i is independent of the choice of the linear rigidification also shows that the global isomorphism α is independent of the choice of the open affine covering. Therefore, this isomorphism $\alpha : \Delta(L)^{\otimes 12} \cong \mathcal{O}_S$ is the desired canonical trivialization of $\Delta(L)^{\otimes 12}$.

4 The class of $\Delta(\overline{L})$ in the arithmetic Picard group $\widehat{\mathrm{Pic}}(S)$

4.1 Arithmetic Adams-Riemann-Roch theorem

In this subsection, we describe the arithmetic Adams-Riemann-Roch theorem that we will use to investigate the class of $\Delta(\overline{L})$ in the arithmetic Picard group $\widehat{\mathrm{Pic}}(S)$. One can see [Roe] for more details.

Let X be a quasi-projective scheme over $\mathbb{Z}$ with smooth generic fibre. Then $X(\mathbb{C})$, the set of complex points of the variety $X \times_{\mathbb{Z}} \mathbb{C}$ admits a structure of complex manifold. Arakelov theory provides a powerful tool in the study of Diophantine geometry by doing algebraic geometry of X over $\mathbb{Z}$ and hermitian complex geometry of $X(\mathbb{C})$ simultaneously. For instance, in the setting of Arakelov geometry, we have hermitian vector bundles on X and the arithmetic Grothendieck group $\widehat{K_0}(X)$ associated to X, which are the main objects in the expression of the arithmetic Adams-Riemann-Roch theorem.

Definition 4.1. Denote by F_∞ the antiholomorphic involution of $X(\mathbb{C})$ induced by the complex conjugation. A hermitian vector bundle $\overline{E}$ on X is an algebraic vector bundle E on X, endowed with a hermitian metric on the associated holomorphic vector bundle $E_{\mathbb{C}}$ on $X(\mathbb{C})$ which is invariant under F_∞.

Denote by $A^{p,p}(X)$ the set of real smooth forms ω of type (p,p) on $X(\mathbb{C})$ which satisfy $F_\infty^*\omega = (-1)^p\omega$, and by $Z^{p,p}(X) \subseteq A^{p,p}(X)$ the kernel of the differential operator $d = \partial + \overline{\partial}$. We shall write $\widetilde{A}(X)$ for the set of form classes

$$\widetilde{A}(X) := \bigoplus_{p\geq 0} \left(A^{p,p}(X)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial})\right)$$

and

$$Z(X) := \bigoplus_{p\geq 0} Z^{p,p}(X).$$

To every hermitian vector bundle $\overline{E}$ on X, we may associate a Chern character form $\mathrm{ch}(\overline{E}) := \mathrm{ch}(E_{\mathbb{C}}, h)$ which is defined by the Chern-Weil theory on hermitian holomorphic vector bundles on complex manifolds. Similarly, we have Todd form $\mathrm{Td}(\overline{E})$. Notice that the Chern-Weil theory is not additive for short exact sequence of hermitian vector bundles. Let $\overline{\varepsilon} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$ be an exact sequence of hermitian vector bundles on X, we can associate to it a Bott-Chern secondary characteristic class $\widetilde{\mathrm{ch}}(\overline{\varepsilon}) \in \widetilde{A}(X)$ which satisfies the differential equation

$$\mathrm{dd}^c\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = \mathrm{ch}(\overline{E}') - \mathrm{ch}(\overline{E}) + \mathrm{ch}(\overline{E}'')$$

where dd^c is the differential operator $\frac{\overline{\partial}\partial}{2\pi i}$.

Definition 4.2. The arithmetic Grothendieck group $\widehat{K_0}(X)$ with respect to X is the abelian group generated by the elements of $\widetilde{A}(X)$ and by the isometry classes of hermitian vector bundles on X, modulo the following relations:

(i). for every exact sequence $\overline{\varepsilon}$ as above, $\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = \overline{E}' - \overline{E} + \overline{E}''$;

(ii). if $\alpha \in \widetilde{A}(X)$ is the sum of two elements α' and α'' in $\widetilde{A}(X)$, then the equality $\alpha = \alpha' + \alpha''$ still holds in $\widehat{K_0}(X)$.

We now recall the definitions of λ-ring and associated Adams operations.

Definition 4.3. A λ-ring is a unitary ring R with operations $\lambda^k, k \in \mathbb{N}$ satisfying the following axioms.

(i). $\lambda^0 = 1$, $\lambda^1(x) = x \quad \forall x \in R$, $\lambda^k(1) = 0 \quad \forall k > 1$.

(ii). $\lambda^k(x+y) = \sum_{i=0}^k \lambda^i(x) \cdot \lambda^{k-i}(y)$.

(iii). $\lambda^k(xy) = P_k(\lambda^1(x), \ldots, \lambda^k(x); \lambda^1(y), \ldots, \lambda^k(y))$ for some universal polynomial P_k with integral coefficients.

(iv). $\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), \ldots, \lambda^{kl}(x))$ for some universal polynomial $P_{k,l}$ with integral coefficients.

Putting $\lambda_t(x) := \sum_k \lambda^k(x) t^k$, we have $\lambda_t(x+y) = \lambda_t(x) \cdot \lambda_t(y)$ by (ii). For the definitions of P_k and $P_{k,l}$, we refer to [SABK, I. 4.2, 4.3]

Given a λ-ring R, the relationship between the Adams operations ψ^k and the λ-operations is the following. Define a formal power series ψ_t by the formula

$$\psi_t(x) := \frac{-t \cdot \mathrm{d}\lambda_{-t}(x)/\mathrm{d}t}{\lambda_{-t}(x)}.$$

The Adams operations are then given by the identity (cf. [SGA6, V, Appendice])

$$\psi_t(x) = \sum_{k \geq 1} \psi^k(x) t^k.$$

Now consider the group $\Gamma(X) := Z(X) \oplus \widetilde{A}(X)$, we equip it with a grading $\Gamma(X) = \oplus_{p \geq 0} \Gamma_p(X)$ where

$$\Gamma_p(X) := \begin{cases} Z^{p,p}(X) \oplus \widetilde{A}^{p-1,p-1}(X), & \text{if } p \geq 1; \\ Z^{0,0}(X), & \text{if } p = 0. \end{cases}$$

We define a bilinear map $*$ from $\Gamma(X) \times \Gamma(X)$ to $\Gamma(X)$ by the formula

$$(\omega, \eta) * (\omega', \eta') = (\omega \wedge \omega', \omega \wedge \eta' + \eta \wedge \omega' + (\mathrm{dd}^c \eta) \wedge \eta').$$

This map endows $\Gamma(X)$ with the structure of a commutative graded $\mathbb{R}$-algebra (cf. [GS, Lemma 7.3.1]). Hence there is a unique λ-ring structure on $\Gamma(X)$ such that the k-th associated Adams operation is given by the formula $\psi^k(x) = \sum_{i \geq 0} k^i x_i$, where x_i stands for the component of degree i of the element $x \in \Gamma(X)$ (cf. [SGA6, 7.2, p. 361]).

Definition 4.4. If $\overline{E} + \eta$ and $\overline{E}' + \eta'$ are two generators of $\widehat{K_0}(X)$, then we may define a product $\otimes$ by the formula

$$(\overline{E} + \eta) \otimes (\overline{E}' + \eta') = \overline{E} \otimes \overline{E}' + [(\mathrm{ch}(\overline{E}), \eta) * (\mathrm{ch}(\overline{E}'), \eta')]$$

where $[\cdot]$ refers to the projection on the second component of $\Gamma(X)$. If $k \geq 0$, we set

$$\lambda^k(\overline{E}+\eta) = \wedge^k(\overline{E}) + [\lambda^k(\mathrm{ch}(\overline{E}),\eta)]$$

where $\lambda^k(\mathrm{ch}(\overline{E}),\eta)$ stands for the image of $(\mathrm{ch}(\overline{E}),\eta)$ under the k-th λ-operation of $\Gamma(X)$.

It was shown by Roessler in [Roe1] that $\widehat{K_0}(X)$ with the product $\otimes$ and the operations λ^k in Definition 4.4 is actually a λ-ring.

Let Y be another quasi-projective scheme over $\mathbb{Z}$ with smooth generic fibre and suppose that $f : X \to Y$ is a flat projective morphism which is smooth over $\mathbb{Q}$. In this situation, $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ is a holomorphic proper submersion between complex manifolds. A Kähler fibration structure on $f_{\mathbb{C}}$ is a real closed $(1,1)$-form ω on $X(\mathbb{C})$ which induces Kähler metrics on the fibres (cf. [BK, Def. 1.1, Thm. 1.2]). If $f_{\mathbb{C}}$ is endowed with a Kähler fibration structure, we may define a reasonable push-forward morphism $f_* : \widehat{K_0}(X) \to \widehat{K_0}(Y)$. For instance, we can fix a conjugation invariant Kähler metric on $X(\mathbb{C})$ and choose corresponding Kähler form ω as the Kähler fibration structure.

Let (E,h^E) be a hermitian vector bundle on X such that E is f-acyclic i.e. the higher direct image R^qf_*E vanishes for $q>0$. By semi-continuity theorem (cf. [Har, Theorem III.12.8, Cor. III.12.9]) the sheaf of module $f_*E := R^0f_*E$ is locally free and the natural map

$$(R^0f_*E)_y \to H^0(X_y, E\mid_{X_y})$$

is an isomorphism for every point $y \in Y$. In particular, we have natural isomorphism

$$(R^0f_*E_{\mathbb{C}})_y \to H^0(X(\mathbb{C})_y, E_{\mathbb{C}}\mid_{X(\mathbb{C})_y})$$

for every point $y \in Y(\mathbb{C})$. On the other hand, we may endow $H^0(X(\mathbb{C})_y, E_{\mathbb{C}}\mid_{X(\mathbb{C})_y})$ with a L^2-hermitian product given by the formula

$$< s,t >_{L^2} := \frac{1}{(2\pi)^{d_y}}\int_{X(\mathbb{C})_y} h^E(s,t)\frac{\omega^{d_y}}{d_y!}$$

where d_y is the complex dimension of the fibre $X(\mathbb{C})_y$. It can be shown that these hermitian products depend on y in a C^∞ manner (cf. [BGV, p.278]) and hence define a hermitian metric on $(f_*E)_{\mathbb{C}}$. This metric is called the L^2-metric. Let $(Tf_{\mathbb{C}},h_f)$ be the relative holomorphic tangent bundle with the metric induced by ω. In [BK, Theorem 3.9], Bismut and Köhler constructed a smooth form $T(\omega,h^E) \in A(Y) = \oplus_{p\geq 0}A^{p,p}(Y)$ satisfying the differential equation

$$\mathrm{dd}^cT(\omega,h^E) = \mathrm{ch}(f_*E_{\mathbb{C}},h^{L^2}) - \int_{X(\mathbb{C})/Y(\mathbb{C})}\mathrm{ch}(E_{\mathbb{C}},h^E)\mathrm{Td}(Tf_{\mathbb{C}},h_f).$$

This smooth form is called the higher analytic torsion form associated to (E,h^E), $f_{\mathbb{C}}$ and ω. Its definition is too long and technic, we can not repeat it here. We just would like to mention that the 0-degree part i.e. the function part of $T(\omega,h^E)$ is the famous Ray-Singer analytic torsion of $\overline{E_{\mathbb{C}}}$ on every fibre $X(\mathbb{C})_y$. The Quillen metric $\|\cdot\|_Q$ on $\det(f_*E)$ is defined by

$$\|\cdot\|_Q^2 = e^{T_0(\omega,h^E)}\cdot\|\cdot\|_{L^2}^2 .$$

Remark 4.5. It was shown in [BFL, Corollary 8.10] that the analytic torsion form $T(\omega, h^E)$ is compatible (up to exact ∂- and $\overline{\partial}$-forms) with any base-change of Kähler fibration. Therefore, the Quillen metric $\|\cdot\|_Q$ on $\det(f_*E)$ and the hermitian line bundle $\Delta(\overline{L})$ are compatible with arbitrary base-change.

Definition 4.6. The push-forward morphism $f_* : \widehat{K_0}(X) \to \widehat{K_0}(Y)$ is defined as follows.

(i). for hermitian holomorphic vector bundle (E, h) on X such that E is f-acyclic, $f_*(E, h) := (f_*E, h^{L^2}) - T(\omega, h^E)$;

(ii). for $\eta \in \widetilde{A}(X)$, $f_*\eta := \int_{X(\mathbb{C})/Y(\mathbb{C})} \mathrm{Td}(Tf_{\mathbb{C}}, h_f)\eta$.

Remark 4.7. f_* is a well-defined group homomorphism and it satisfies the projection formula.

The next important object appearing in the expression of the Adams-Riemann-Roch theorem is the following R-genus.

Definition 4.8. The R-genus is the unique additive characteristic class defined for a line bundle L by the formula

$$R(L) = \sum_{m \text{ odd}, \geq 1} \big(2\zeta'(-m) + \zeta(-m)(1 + \frac{1}{2} + \cdots + \frac{1}{m})\big)\frac{\mathrm{c}_1(L)^m}{m!}$$

where $\zeta(s)$ is the Riemann zeta-function.

To state the arithmetic Adams-Riemann-Roch theorem, we still need the Bott's cannibalistic classes. For any λ-ring R, denote by R_{fin} its subset of elements of finite λ-dimension. For each $k \geq 1$, the Bott's cannibalistic class θ^k is uniquely determined by the following properties

(i). θ^k maps R_{fin} into R_{fin} and the equation $\theta^k(a+b) = \theta^k(a)\theta^k(b)$ holds for all $a, b \in R_{\mathrm{fin}}$;

(ii). θ^k is functorial with respect to λ-ring morphisms;

(iii). if e is a line element (its λ-dimension is 1), then $\theta^k(e) = \sum_{i=0}^{k-1} e^i$.

Now, consider the graded commutative group $\widetilde{A}(X) = \oplus_{p\geq 0}\widetilde{A}^{p,p}(X)$, giving degree p to differential forms of type (p, p). We define $\phi^k(\omega) = \sum_{i=0}^{\infty} k^i\omega_i$ where ω_i is the component of degree i of $\omega \in \widetilde{A}(X)$. Then one can compute that $\psi^k(\omega) = k \cdot \phi^k(\omega)$ where on the left hand side ω is regarded as an element of the λ-ring $\Gamma(X)$.

Let $\overline{E}$ be a hermitian vector bundle on X, then the form $k^{-\mathrm{rk}(E)}\mathrm{Td}^{-1}(\overline{E})\phi^k(\mathrm{Td}(\overline{E}))$ is by construction a universal polynomial in the Chern forms $c_i(\overline{E})$. The associated symmetric polynomial in $r = \mathrm{rk}(E)$ variables is denoted by CT^k, and one can compute that

$$CT^k = k^r \prod_{i=1}^{r} \frac{e^{T_i} - 1}{T_i e^{T_i}} \cdot \frac{kT_i e^{kT_i}}{e^{kT_i} - 1}$$

where $T_1, \ldots, T_r$ are the variables. For an exact sequence of hermitian holomorphic vector bundles $\overline{\varepsilon} : 0 \to \overline{E'} \to \overline{E} \to \overline{E''} \to 0$ on a complex manifold, the Bott-Chern secondary characteristic class associated to $\overline{\varepsilon}$ and to CT^k will be denoted by $\widetilde{\theta}^k(\overline{\varepsilon})$.

We now turn back to the flat projective morphism $f : X \to Y$, which is smooth over $\mathbb{Q}$. Suppose that f is a local complete intersection morphism. Let $i : X \to P$ be a regular immersion and $p : P \to Y$ be a smooth morphism, such that $f = p \circ i$. Endow P with a Kähler metric and the normal bundle $N_{P/X}$ with some hermitian metric. Denote by $\overline{\mathcal{N}}$ be the exact sequence $0 \to \overline{Tf_{\mathbb{C}}} \to \overline{TP_{\mathbb{C}}} \to \overline{N}_{P(\mathbb{C})/X(\mathbb{C})} \to 0$.

Definition 4.9. The arithmetic Bott class $\theta^k(\overline{Tf}^{\vee})^{-1}$ of f is the element $\theta^k(\overline{N}_{P/X}^{\vee})\widetilde{\theta}^k(\overline{\mathcal{N}}) + \theta^k(\overline{N}_{P/X}^{\vee})\theta^k(i^*\overline{TP}^{\vee})^{-1}$ in $\widehat{K_0}(X)[1/k]$.

Remark 4.10. The arithmetic Bott class of f depends neither on i nor on the metrics on P and on $N_{P/X}$ (cf. [Roe, Lemma 3.5]).

Theorem 4.11. *(arithmetic Adams-Riemann-Roch) Let $f : X \to Y$ be as above. For each $k \geq 1$, let $\theta_A^k(\overline{Tf}^{\vee})^{-1} = \theta^k(\overline{Tf}^{\vee})^{-1} \cdot (1 + R(Tf_{\mathbb{C}}) - k \cdot \phi^k(R(Tf_{\mathbb{C}})))$. Then for the map $f_* : \widehat{K_0}(X)[1/k] \to \widehat{K_0}(Y)[1/k]$, the equality*

$$\psi^k(f_*(x)) = f_*(\theta_A^k(\overline{Tf}^{\vee})^{-1} \cdot \psi^k(x))$$

holds in $\widehat{K_0}(Y)[1/k]$ for all $k \geq 1$ and $x \in \widehat{K_0}(X)[1/k]$.

Proof. This is [Roe, Theorem 3.6]. □

4.2 The γ-filtration of arithmetic K_0-theory

Let X be a quasi-projective scheme over $\mathbb{Z}$ with smooth generic fibre as in last subsection. In this subsection, we shall recall the γ-filtration of the λ-ring $\widehat{K_0}(X)$ and prove some basic facts.

Recall that the γ-operations on a λ-ring are defined by the formula

$$\gamma_t(x) = \sum_{i\geq 0} \gamma^i(x)t^i := \lambda_{t/(1-t)}(x).$$

By construction, the γ^i also define a pre-λ-ring structure on $\widehat{K_0}(X)$: that is, for all positive integers k we have $\gamma^0(x) = 1$, $\gamma^1(x) = x$ and

$$\gamma^k(x+y) = \sum_{i=0}^{k} \gamma^i(x)\gamma^{k-i}(y).$$

Moreover, it follows from the definition that if u is the class of a hermitian line bundle on X, then $\gamma_t(u-1) = 1 + (u-1)t$ and $\gamma_t(1-u) = \sum_{i\geq 0}(1-u)^i t^i$. This implies that $\gamma^i(u-1) = 0$ for $i > 1$ and $\gamma^i(1-u) = (1-u)^i$ for $i \geq 0$.

Now, for any generator $(\overline{E}, \eta)$ of $\widehat{K_0}(X)$, define $\varepsilon(\overline{E}, \eta) = \mathrm{rk}(E)$. This map extends to an augmentation on $\widehat{K_0}(X)$, namely a λ-ring homomorphism from $\widehat{K_0}(X)$ to $\mathbb{Z}$. We then construct the γ-filtration $F^n\widehat{K_0}(X)(n \geq 0)$ of $\widehat{K_0}(X)$ as follows. For $n = 0$, $F^0\widehat{K_0}(X) :=$

$\widehat{K_0}(X)$, for $n=1$, $F^1\widehat{K_0}(X) := \ker(\varepsilon)$ and for $n \geq 2$, $F^n\widehat{K_0}(X)$ is defined to be the additive subgroup generated by the elements $\gamma^{r_1}(x_1)\cdots\gamma^{r_s}(x_s)$, where $x_1,\ldots,x_s \in F^1\widehat{K_0}(X)$ and $\sum_{i=1}^s r_i \geq n$. Therefore $F^0\widehat{K_0}(X) \supseteq F^1\widehat{K_0}(X) \supseteq F^2\widehat{K_0}(X) \supseteq \cdots$ and it is easily checked that the $F^n\widehat{K_0}(X)$ are ideals that form a ring filtration. We shall denote by $\mathrm{Gr}^i\widehat{K_0}(X)$ the quotient group $F^i\widehat{K_0}(X)/F^{i+1}\widehat{K_0}(X)$.

Proposition 4.12. *Let $j \geq 1$ and let $n \geq 0$ be an integer. If $x \in F^n\widehat{K_0}(X)$, then*

$$\psi^j(x) = j^n x \mod F^{n+1}\widehat{K_0}(X).$$

Proof. This statement is actually correct for any augmented λ-ring, see the last lemma in [RSS, p. 96]. □

Definition 4.13. The γ-filtration of an augmented λ-ring R is called locally nilpotent, if for every $x \in F^1R$, there exists a number $N(x) \in \mathbb{N}$, depending on x, such that $\gamma^{r_1}(x)\cdots\gamma^{r_d}(x) = 0$ whenever $\sum_{i=1}^d r_i > N(x)$. It is called nilpotent, if there exists a number $N \in \mathbb{N}$, such that $F^nR = 0$ for all $n > N$.

It was shown by Roessler in [Roe, Prop. 4.5] that the γ-filtration of $\widehat{K_0}(X)$ is locally nilpotent, hence $\widehat{K_0}(X)$ fulfills the conditions in the following proposition.

Proposition 4.14. *Let R be an augmented λ-ring with locally nilpotent γ-filtration. Then for any $n \geq 0$,*

$$F^nR_{\mathbb{Q}} = \bigoplus_{i=n}^{\infty} V_i$$

where V_i is the k^i-eigenspace of ψ^k on $R_{\mathbb{Q}}$, $k > 1$, and V_i does not depend on k.

Proof. This is in complete analogy to the proof of [RSS, p. 97, Theorem 1]. □

Corollary 4.15. *If $n > \dim(X)$, then $F^n\widehat{K_0}(X)_{\mathbb{Q}} = 0$.*

Proof. Firstly notice that the γ-filtration of the algebraic Grothendieck group $K_0(X)$ is nilpotent, precisely $F^nK_0(X) = 0$ whenever n is greater than the dimension of X. By construction the forgetful map $\widehat{K_0}(X) \to K_0(X)$ is a λ-ring morphism, then any element $x \in F^n\widehat{K_0}(X)$ is represented by a smooth form η if $n > \dim X$. We claim that $\eta = 0$ in $F^n\widehat{K_0}(X)_{\mathbb{Q}}$, which implies the statement in this corollary. Indeed, write $\eta = \sum_{i\geq 0}\eta^{(i)}$ where $\eta^{(i)}$ is the i-th component of η, by the definition of the λ-ring structure, we know that $\eta^{(i)} \in V_{i+1}$. Since $\eta^{(i)} = 0$ when $i > \dim X(\mathbb{C})$, we have $\eta \in \bigoplus_{i=1}^{\dim X} V_i$. Then our claim follows from Proposition 4.14 □

Now, we introduce a truncated arithmetic Chern character

$$\widehat{\mathrm{ch}} : \widehat{K_0}(X)[1/k] \to \mathbb{Z}[1/k] \oplus \widehat{\mathrm{Pic}}(X)[1/k].$$

This Chern character is an abelian group homomorphism defined as follows: (i). for a hermitian vector bundle $\overline{E}$ on X, $\widehat{\mathrm{ch}}(\overline{E}/k^t) = (\mathrm{rk}(E)/k^t, \det(\overline{E})^{1/k^t})$; (ii). for an element $\omega \in \widetilde{A}(X)$, $\widehat{\mathrm{ch}}(\omega/k^t) = (0, (\mathcal{O}_X, \omega)^{1/k^t})$ where $(\mathcal{O}_X, \omega)$ stands for the trivial bundle with the metric given by $\| 1 \|^2 = e^{-\omega_0}$. By using [GS, Prop. 1.2.5], one can immediately check that this definition is compatible with the generating relation of $\widehat{K_0}(X)$. Moreover, let us introduce the paring

$$(r_1/k^{t_1}, m_1^{1/k^{l_1}}) \bullet (r_2/k^{t_2}, m_2^{1/k^{l_2}}) := (r_1r_2/k^{t_1+t_2}, m_2^{r_1/k^{t_1+l_2}} \otimes m_1^{r_2/k^{t_2+l_1}})$$

in the group $\mathbb{Z}[1/k] \oplus \widehat{\mathrm{Pic}}(X)[1/k]$. The paring $\bullet$ makes this group into a commutative ring. It can be shown that the arithmetic Chern character is a ring homomorphism, by the properties of the determinant and by the definition of the arithmetic Grothendieck group.

In particular, by composing with the projection to the second factor, we get a group homomorphism

$$\det : \widehat{K_0}(X) \to \widehat{\mathrm{Pic}}(X).$$

The main result of this subsection is the following.

Theorem 4.16. *The morphism* det *induces an isomorphism* $\mathrm{Gr}^1(\widehat{K_0}(X)) \cong \widehat{\mathrm{Pic}}(X)$.

To prove this theorem, we need the following lemmas.

Lemma 4.17. *Let $x \in \widehat{K_0}(X)$ which is represented by a smooth form η. Then for any $k \geq 2$, the function part of $\gamma^k(\eta)$ vanishes.*

Proof. It is well known that the λ-operations λ^i and corresponding Adams operations ψ^k are related by the following Newton formula

$$\psi^k(x) - \lambda^1(x)\psi^{k-1}(x) + \cdots + (-1)^{k-1}\lambda^{k-1}(x)\psi^1(x) = (-1)^{k+1}k\lambda^k(x).$$

Then by the construction of the ring structure of $\Gamma(X)$, the function part of $\lambda^k(x)$ is $(-1)^{k+1}\eta^{(0)}$. Next, we know that the relation between the γ-operations and λ-operations is

$$\gamma^k(x) = \sum_{j=1}^{k} \binom{k-1}{j-1} \lambda^j(x).$$

Then our lemma follows from the combinatorial identity

$$\sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{j+1} = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j = (1-1)^{k-1} = 0.$$

□

Remark 4.18. Actually, we conjecture that for any $k \geq 2$, the i-th component of $\gamma^k(\eta)$ vanishes if $i < k-1$.

Lemma 4.19. *Let $x \in \widehat{K_0}(X)$ which is represented by a smooth form η and suppose that the function part of η vanishes, then $x \in F^2\widehat{K_0}(X)$.*

Proof. We claim that there exists an element $\omega \in \widetilde{A}(X)$ such that $\gamma^2(\omega) = \eta$. Indeed, we need to solve the equation $\omega + \lambda^2(\omega) = \eta$ which is equivalent to $\omega + \frac{1}{2}\omega \wedge \mathrm{dd}^c\omega - \frac{1}{2}\psi^2(\omega) = \eta$ by the Newton formula. But taking $\omega^{(0)}$ to be any real valued smooth function (eg. the constant function 0) the above equation can be certainly solved by solving $\omega^{(i)}$ $(i \geq 1)$ one by one. This process will terminate after finitely many steps because the dimension of $X(\mathbb{C})$ is finite. □

Proof. (of Theorem 4.16) We firstly prove that for any $x \in F^2\widehat{K_0}(X)$ we have $\det(x) = 1$, then the morphism $\det : F^1\widehat{K_0}(X)/F^2\widehat{K_0}(X) \to \widehat{\mathrm{Pic}}(X)$ is well-defined. If x is the product of two generators of $\ker(\varepsilon)$, i.e. if $x = \gamma^1(\overline{E}_1 - \overline{F}_1 + \eta_1)\gamma^1(\overline{E}_2 - \overline{F}_2 + \eta_2)$ where $\overline{E}_i, \overline{F}_i$ are hermitian vector bundles on X such that $\mathrm{rk}(E_i) = \mathrm{rk}(F_i)$ and $\eta_i \in \widetilde{A}(X)$ $(i = 1, 2)$, then it is readily checked that $\widehat{\mathrm{ch}}(x) = (0, 1)$ and hence $\det(x) = 1$ in $\widehat{\mathrm{Pic}}(X)$. Notice that $\widehat{\mathrm{ch}}$ is a ring homomorphism, the γ^k define a pre-λ-ring structure on $\widehat{K_0}(X)$ and the function part of $\gamma^k(\eta)$ vanishes (by Lemma 4.17), we may reduce our proof to the case where $x = \gamma^i(\overline{E} - \overline{F})$ with $\mathrm{rk}(E) = \mathrm{rk}(F)$ and $i \geq 2$. By the splitting principle (cf. [Roe1, Theorem 4.1]), we may assume that $\mathrm{rk}(E) = \mathrm{rk}(F) = 1$, then $x = \gamma^i((\overline{E} - 1) + (1 - \overline{F}))$ and we may furthermore reduce the proof to the case $x = \gamma^i(1 - \overline{F})$ with $i \geq 2$ since $\gamma^i(\overline{E} - 1) = 0$ for $i > 1$. In this case, the statement that $\det(x) = 1$ is correct because $\gamma^i(1 - \overline{F}) = (1 - \overline{F})^i$ for $i \geq 0$.

Now, let g be a map from $\widehat{\mathrm{Pic}}(X)$ to $\mathrm{Gr}^1(\widehat{K_0}(X))$ which sends $\overline{L}$ to $\overline{L} - 1 \bmod F^2\widehat{K_0}(X)$. This is a group homomorphism because

$$(\overline{LL}' - 1) - (\overline{L} - 1) - (\overline{L}' - 1) = (\overline{L} - 1)(\overline{L}' - 1)$$

which is an element in $F^2\widehat{K_0}(X)$. Moreover, since $\det(\overline{L} - 1) = \det(\overline{L}) = \overline{L}$, we have $\det \circ g = \mathrm{Id}$. This implies that $\det : \mathrm{Gr}^1(\widehat{K_0}(X)) \to \widehat{\mathrm{Pic}}(X)$ is surjective.

Finally, we prove that det is injective. Let $x \in F^1\widehat{K_0}(X)$, if x is represented by a smooth form η, then by [GS, Prop. 1.2.5] we have $g \circ \det(\eta) = \eta^{(0)} \bmod F^2\widehat{K_0}(X)$. But $\eta^{(0)} = \eta \bmod F^2\widehat{K_0}(X)$ by Lemma 4.19, so $g \circ \det(\eta) = \mathrm{Id}$ for elements in $\widetilde{A}(X)$. Next, we assume that $x = \overline{E} - \overline{F}$ such that $\mathrm{rk}(E) = \mathrm{rk}(F)$. By the splitting principle, we can write $x = \sum n_i\overline{L_i} = \sum n_i(\overline{L_i} - 1)$ where $n_i \in \mathbb{Z}, \sum n_i = 0$ and $\overline{L_i}$ are hermitian line bundles on X. Then $\det(x) = \prod \overline{L_i}^{n_i}$ and $g \circ \det(x) = x \bmod F^2\widehat{K_0}(X)$. □

4.3 The class of $\Delta(\overline{L})$ in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})$

In this subsection, we shall complete the proof of our main theorem, Theorem 1.3. Let $\pi : \widetilde{Z}_{g,d,1} \to \widetilde{H}_{g,d,1}$ be the universal abelian scheme of the moduli functor $\widetilde{\mathcal{H}}_{g,d,1}$ with the universal rigidified relatively ample line bundle L. As an arithmetic extension of Section 3., it is clear that we only need to show that there exist some $\mathbf{PGL}_N$-invariant hermitian metric on L and some $\mathbf{PGL}_N$-invariant Kähler fibration structure on $\pi_{\mathbb{C}}$ such that $\Delta(\overline{L})$ has a torsion class in the

arithmetic Picard group $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})$ and the bound of its order can be chosen to be independent of L.

Let $e : \widetilde{H}_{g,d,1} \to \widetilde{Z}_{g,d,1}$ be the unit section and let $\eta : e^*L \cong \mathcal{O}_{\widetilde{H}_{g,d,1}}$ be the universal rigidification of L. Denote by $\gamma : [-1]^*L \cong L$ the isomorphism which is compatible with the rigidification. In [MB, Prop. 2.1, p. 48], Moret-Bailly shows that any line bundle on an abelian variety A over $\mathbb{C}$ admits a unique hermitian metric such that its curvature form is translation invariant and such that it is compatible with the rigidification. Moret-Bailly's proof relies on the cubical structure on L, which is an isomorphism

$$\alpha : \mathcal{D}_3(L) := \bigotimes_{\emptyset \neq I \subset \{1,2,3\}} (m_I^* L)^{\otimes (-1)^{\mathrm{Card}(I)}} \longrightarrow \mathcal{O}_{A\times A\times A}$$

satisfying some symmetry and cocycle conditions (cf. [MB, Definition 2.4.5, p. 19]), here $m_I : A\times A\times A$ is the morphism $(x_1, x_2, x_3) \mapsto \Sigma_{i\in I} x_i$. Endowing $\mathcal{O}_{A\times A\times A}$ with the trivial metric and endowing $\mathcal{D}_3(L)$ with the hermitian metric such that α is an isometry. Moret-Bailly actually proves that the metric on $\mathcal{D}_3(L)$ obtained in this way arises from a unique hermitian metric on L. This argument is also valid for the relative setting. That means, to $(\widetilde{Z}_{g,d,1}/\widetilde{H}_{g,d,1}, L)$, there exists a unique metric on $L_{\mathbb{C}}$ such that the first Chern form $\mathrm{c}_1(\overline{L})$ is translation invariant on the fibres and such that η is an isometry. We endow $L_{\mathbb{C}}$ with this metric. By unicity, this metric is $\mathbf{PGL}_N$-invariant, because the $\mathbf{PGL}_N$-action only changes the linear rigidification, it doesn't affect the structures of projective abelian scheme and of the rigidified relatively ample line bundle. Also by the unicity of this metric, $\gamma : [-1]^*\overline{L} \cong \overline{L}$ is an isometry. Moreover, by construction, this metric is compatible with the theorem of the cube, so we have isometry $[k]^*\overline{L} \cong \overline{L}^{k^2}$ for any $k \geq 1$. Next, notice that the real $(1,1)$-form $\mathrm{c}_1(\overline{L})$ is positive on every fibre because L is a relatively ample line bundle. Then $\mathrm{c}_1(\overline{L})$ defines a hermitian metric on the relative tangent bundle $T\pi$, and hence a Kähler fibration structure on $\pi_{\mathbb{C}}$ (cf. [BK, Theorem 1.2]). This Kähler fibration structure is $\mathbf{PGL}_N$-invariant because the metric on L is so. We endow Ω_π and ω_π with the metrics induced by the metric on $T\pi$. Finally, since $\mathrm{c}_1(\overline{L})$ is translation invariant on the fibres, we have a canonical isometry $\pi^* e^* \overline{\Omega}_\pi \cong \overline{\Omega}_\pi$.

We shall use the arithmetic Adams-Riemann-Roch theorem to prove that there exist a positive integer m and an isometry $\Delta(\overline{L})^m \cong \overline{\mathcal{O}}_{\widetilde{H}_{g,d,1}}$. We compute in $\widehat{K_0}(\widetilde{H}_{g,d,1})[1/k]$:

$$\begin{aligned}
\psi^{k^2}(\pi_*\overline{L}) &= \pi_*(\theta_A^{k^2}(\overline{\Omega}_\pi)^{-1} \cdot \psi^{k^2}(\overline{L})) \\
&= \pi_*(\theta_A^{k^2}(\overline{\Omega}_\pi)^{-1} \overline{L}^{k^2}) \\
&= \pi_*(\theta_A^{k^2}(\overline{\Omega}_\pi)^{-1} [k]^*\overline{L}) \\
&= \pi_*([k]^*\overline{L}) \theta_A^{k^2}(e^*\overline{\Omega}_\pi)^{-1}.
\end{aligned}$$

In other words, we have the identity

$$\theta_A^{k^2}(e^*\overline{\Omega}_\pi)\psi^{k^2}(\pi_*\overline{L}) = \pi_*([k]^*\overline{L})$$

which holds in $\widehat{K_0}(\widetilde{H}_{g,d,1})[1/k]$. We now apply the arithmetic Chern character $\widehat{\mathrm{ch}}$ to the above identity.

To simplify the expression, we shall replace the multiplicative notation $\otimes$ by the additive notation $+$ in the group $\widehat{\mathrm{Pic}}$. Moreover, by the splitting principle we may suppose that $e^*\overline{\Omega}_\pi = \overline{Q}_1 + \cdots + \overline{Q}_g$ in $\widehat{K}_0(\widetilde{H}_{g,d,1})$, where $\overline{Q}_1, \ldots, \overline{Q}_g$ are hermitian line bundles. So we have

$$\begin{aligned}\widehat{\mathrm{ch}}(\theta_A^{k^2}(e^*\overline{\Omega}_\pi)) &= \widehat{\mathrm{ch}}(\theta^{k^2}(e^*\overline{\Omega}_\pi)) \\ &= (k^2 + \frac{k^2(k^2-1)}{2}\det(\overline{Q}_1)) \bullet \cdots \bullet (k^2 + \frac{k^2(k^2-1)}{2}\det(\overline{Q}_g)) \\ &= k^{2g} + \frac{k^2(k^2-1)k^{2g-2}}{2}\det(e^*\overline{\Omega}_\pi).\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathrm{ch}}(\theta_A^{k^2}(e^*\overline{\Omega}_\pi)) \bullet \widehat{\mathrm{ch}}(\psi^{k^2}(\pi_*\overline{L})) &= (k^{2g} + \frac{k^2(k^2-1)k^{2g-2}}{2}\det(e^*\overline{\Omega}_\pi)) \bullet (d + k^2\det{}_Q(\pi_*\overline{L})) \\ &= k^{2g}d + k^{2g+2}\det{}_Q(\pi_*\overline{L}) + \frac{dk^2(k^2-1)k^{2g-2}}{2}\det(e^*\overline{\Omega}_\pi).\end{aligned}$$

On the other hand, we have

$$\widehat{\mathrm{ch}}(\pi_*([k]^*\overline{L}) = dk^{2g} + \det{}_Q(\pi_*([k]^*\overline{L})).$$

This follows from the fact that the degree of the isogeny $[k]$ on $\widetilde{Z}_{g,d,1}$ is k^{2g} and the fact that the rank of $\pi_*([k]^*L)$ is dk^{2g} (cf. [Mum, Theorem 2, p. 121]). Finally, multiplying by k^{-2g} and specializing to $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})[1/k]$, we get an identity

$$k^2\det{}_Q(\pi_*\overline{L}) + \frac{d(k^2-1)}{2}\det(e^*\overline{\Omega}_\pi) = k^{-2g}\det{}_Q(\pi_*([k]^*\overline{L}))$$

in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})[1/k]$.

We now compare $\det_Q(\pi_*\overline{L})$ with $\det_Q(\pi_*([k]^*\overline{L}))$, we need the following lemmas.

Lemma 4.20. *Let $\overline{\varepsilon}$ be the exact sequence $0 \to 0 \to \overline{T\pi} \to [k]^*\overline{T\pi} \to 0$ of hermitian vector bundles on $\widetilde{Z}_{g,d,1}(\mathbb{C})$, and let η be the smooth form $\widetilde{\mathrm{Td}}(\overline{\varepsilon})\mathrm{Td}^{-1}(\overline{T\pi})$. Then for any positive integer l, the identity $\psi^l([k]_*\overline{O}_{\widetilde{Z}_{g,d,1}} - [k]_*\eta) = [k]_*\overline{O}_{\widetilde{Z}_{g,d,1}} - [k]_*\eta$ holds in $\widehat{K}_0(\widetilde{Z}_{g,d,1})[1/l]$.*

Proof. We apply the arithmetic Adams-Riemann-Roch theorem to the isogeny $[k]$ which is étale over $\mathbb{C}$. The point is to compute $\theta_A^l(\overline{T[k]}^\vee)^{-1}$ which is $\theta^l(\overline{T[k]}^\vee)^{-1}$ since the relative tangent bundle of $[k]_\mathbb{C}$ vanishes. This computation can be done by using a $\theta^l(\cdot)^{-1}$-version of [GS1, Prop. 1. (ii), p. 504], an essentially same reasoning shows that

$$\theta^l(\overline{T[k]}^\vee)^{-1} = \theta^l(\overline{T\pi}^\vee)^{-1} \cdot [k]^*\theta^l(\overline{T\pi}^\vee) - [k]^*\theta^l(\overline{T\pi}^\vee) \cdot \widetilde{\theta}^l(\overline{\varepsilon}).$$

Following from the fact that $\pi^*e^*\overline{\Omega}_\pi \cong \overline{\Omega}_\pi$ and that $\pi \circ [k] = \pi$ we obtain

$$\theta^l(\overline{T[k]}^\vee)^{-1} = 1 - \theta^l(\overline{T\pi}^\vee) \cdot \widetilde{\theta}^l(\overline{\varepsilon}).$$

Next, by [Roe, Lem. 6.11, Prop. 7.3], we have

$$\mathrm{ch}(\theta^l(\overline{T\pi}^\vee)) = l^g \mathrm{Td}(\overline{T\pi})\phi^l(\mathrm{Td}^{-1}(\overline{T\pi}))$$

and

$$\begin{aligned}\widetilde{\theta}^l(\overline{\varepsilon}) &= [l^{-g}\mathrm{Td}^{-1}(\overline{T\pi})\phi^l(\mathrm{Td}(\overline{T\pi}))l^g\widetilde{\mathrm{Td}}(\overline{\varepsilon}) - l\phi^l(\widetilde{\mathrm{Td}}(\overline{\varepsilon}))]l^{-g}\mathrm{Td}^{-1}(\overline{T\pi}) \\ &= l^{-g}\mathrm{Td}^{-2}(\overline{T\pi})\phi^l(\mathrm{Td}(\overline{T\pi}))\widetilde{\mathrm{Td}}(\overline{\varepsilon}) - l^{1-g}\mathrm{Td}^{-1}(\overline{T\pi})\phi^l(\widetilde{\mathrm{Td}}(\overline{\varepsilon}))\end{aligned}$$

So we finally have $\theta^l(\overline{T\pi}^\vee)\cdot\widetilde{\theta}^l(\overline{\varepsilon}) = \mathrm{ch}(\theta^l(\overline{T\pi}^\vee))\widetilde{\theta}^l(\overline{\varepsilon})$ which is nothing but $\eta - \psi^l(\eta)$. This implies that $\psi^l([k]_*\overline{O}_{\widetilde{Z}_{g,d,1}}) = [k]_*\overline{O}_{\widetilde{Z}_{g,d,1}} - [k]_*(\eta - \psi^l(\eta))$ which holds in $\widehat{K_0}(\widetilde{Z}_{g,d,1})[1/l]$. Notice that $\mathrm{dd}^c\widetilde{\mathrm{Td}}(\overline{\varepsilon}) = 0$ and hence $\mathrm{dd}^c\eta = 0$, then $\psi^l([k]_*\eta) = [k]_*(\psi^l(\eta))$. This also can be seen from the fact that the i-th component of $[k]_*\eta$ is $[k]_*(\eta^{(i)})$. Removing $[k]_*(\psi^l(\eta))$ to the left-hand side, we are done. □

Lemma 4.21. *Let η be the smooth form defined in Lemma 4.20. The element $[k]_*\overline{O}_{\widetilde{Z}_{g,d,1}}$ is equal to $k^{2g} + [k]_*\eta$ in $\widehat{K_0}(\widetilde{Z}_{g,d,1})_{\mathbb{Q}}$.*

Proof. Write $x := [k]_*\overline{O}_{\widetilde{Z}_{g,d,1}} - k^{2g} - [k]_*\eta$, then $x \in F^1\widehat{K_0}(\widetilde{Z}_{g,d,1})$. Take any integer $l > 1$, we have $\psi^l(x) - lx \in F^2$ by Proposition 4.12. Then $(1-l)x \in F^2\widehat{K_0}(\widetilde{Z}_{g,d,1})[1/l]$ by Lemma 4.20. Repeating such approach by using Proposition 4.12 and Lemma 4.20, for any positive integer $n > 0$, we get a polynomial

$$P_n(X) := \prod_{i=1}^{n}(1 - X^i)$$

such that $P_n(l) \neq 0$ and $P(l)x \in F^{n+1}\widehat{K_0}(\widetilde{Z}_{g,d,1})[1/l]$. Taking n to be sufficiently large, we deduce our lemma from Corollary 4.15. □

Lemma 4.22. *For any element $x \in \widehat{K_0}(\widetilde{Z}_{g,d,1})$, we have $\pi_*(x) - \pi_*[k]_*(x) = -\pi_*(x\cdot\eta)$ in $\widehat{K_0}(\widetilde{H}_{g,d,1})$.*

Proof. If x is represented by a smooth form, then the statement is clearly true. So we may suppose that x is represented by a π-acyclic hermitian vector bundle $\overline{E}$. In this case, notice that $[k]$ is a finite morphism, so $\overline{E}$ is $[k]$-acyclic. Then the desired identity for $x = \overline{E}$ follows from [Ma, (0.5), p. 543]. □

Proposition 4.23. *The identity $\det_Q(\pi_*([k]^*\overline{L})) = k^{2g}\det_Q(\pi_*\overline{L})$ holds in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})_{\mathbb{Q}}$.*

Proof. Using Lemma 4.21, Lemma 4.22 and the fact that $\mathrm{dd}^c\eta = 0$, we compute

$$\begin{aligned}\pi_*([k]^*\overline{L}) &= \pi_*[k]_*([k]^*\overline{L}\otimes\overline{O}_{\widetilde{Z}_{g,d,1}}) - \pi_*([k]^*\overline{L}\cdot\eta) \\ &= \pi_*(\overline{L}\otimes[k]_*\overline{O}_{\widetilde{Z}_{g,d,1}}) - \pi_*[k]_*([k]^*\overline{L}\cdot\eta) \\ &= \pi_*(k^{2g}\cdot\overline{L}) + \pi_*(\overline{L}\cdot[k]_*\eta) - \pi_*(\overline{L}\cdot[k]_*\eta) \\ &= k^{2g}\cdot\pi_*\overline{L}\end{aligned}$$

which holds in $\widehat{K_0}(\widetilde{H}_{g,d,1})_{\mathbb{Q}}$. Applying the morphism $\widehat{\det}$ to both two sides, we get the desired identity. $\square$

Thanks to Proposition 4.23, we finally conclude that

$$(k^2-1)\cdot \det_Q(\pi_*\overline{L}) + \frac{d(k^2-1)}{2}\cdot \det(e^*\overline{\Omega}_\pi)) = 0$$

in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})_{\mathbb{Q}}$. In other words, $\Delta(\overline{L}) = 0$ in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})_{\mathbb{Q}}$, which means that $\Delta(\overline{L})$ has a torsion class in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})$.

We claim that the bound of the order of $\Delta(\overline{L})$ in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})$ can be chosen to be independent of L. Indeed, Lemma 4.21 has nothing to do with L, then we may choose positive integer n_k (only depends on k) such that

$$n_k\cdot(k^2-1)\cdot \det_Q(\pi_*\overline{L}) + \frac{d(k^2-1)}{2}\cdot \det(e^*\overline{\Omega}_\pi)) = 0$$

in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})[1/k]$. This means $\Delta(\overline{L})^{n_k(k^2-1)}$ is a k^∞-torsion in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})$. Let $k=2$, we see that $\Delta(\overline{L})^{3n_2}$ is a 2^∞-torsion in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})$. Let $k=3$, we see that $\Delta(\overline{L})^{8n_3}$ is a 3^∞-torsion in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})$. Hence $\Delta(\overline{L})^{24n_2n_3}$ is actually trivial in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})$.

Remark 4.24. According to the same reasoning as above, an explicit bound of the order of $\Delta(\overline{L})$ in $\widehat{\mathrm{Pic}}(\widetilde{H}_{g,d,1})$ can be determined if one can show that $F^n\widehat{K_0}(\widetilde{Z}_{g,d,1})$ vanishes for an effective sufficiently large number n.

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